הרצאה 11

נגזרות חלקיות מסדר גבוה

$$f: U \to \mathbb{R}^m$$
$$U \subset_{\circ} \mathbb{R}^n$$
$$\frac{\partial f}{\partial x_{i_1}}(x), x \in U, 1 \le i_1 \le n$$

נניח כי קיימת $\frac{\partial}{\partial x_{i_2}}\left(\frac{\partial f}{\partial x_{i_1}}\right)(x), x \in U$

נניח שקיימת $\frac{\partial}{\partial x_{i_{r-1}}}\left(\frac{\partial}{\partial x_{i_{r-2}}}\left(\dots\left(\frac{\partial f}{\partial x_{i_1}}\right)\right)\right)(x), x \in U$

$$1 \le i_1, \dots, i_{r-1} \le n$$

נניח שקיימת $\frac{\partial}{\partial x_{i_r}}\left(\frac{\partial}{\partial x_{i_{r-1}}}\left(\frac{\partial}{\partial x_{i_{r-2}}}\left(\dots\left(\frac{\partial f}{\partial x_{i_1}}\right)\right)\right)\right)(a), a \in U$.

דוגמא

$$f(x,y) = f(x) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & x^2+y^2 \ne 0 \\ 0 & x = y = 0 \end{cases}$$

$$a = (0,0)$$
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a) \stackrel{?}{=} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a) = \lim_{x\to 0}\frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x}$$

$$\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy}f(0,y)|_{y=0} = 0$$

$$\frac{\partial f}{\partial y}(x,0) = \lim_{y\to 0}\frac{f(x,y)-f(x,0)}{y} = \lim_{y\to 0}\frac{xy\frac{x^2-y^2}{x^2+y^2}}{y} = x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{x\to 0}\frac{x-0}{x} = 1$$
$$f(x,y) = -f(y,x)$$
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = -1$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a) \ne \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

משפט

$$U \subset\circ \mathbb{R}^2 \quad f: U \to \mathbb{R}$$

נניח כי:

1) $\exists \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)$

2) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y), \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)$ רציפות בU.

אזי $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)$

הוכחה

$$\Delta^2 = f(a+h,b+h) - f(a,b+h) - f(a+h,b) + f(a,b) =$$
$$= [f(a+h,b+h) - f(a,b+h)] - [f(a+h,b) - f(a,b)]$$

נגדיר $\varphi(t) \coloneqq f(a+h,t) - f(a,b)$

$$\Delta^2 = \varphi(b+h) - \varphi(b) = (Lagrange) = \frac{d\varphi}{dt}\left(b + \underbrace{\theta_1}_{0<\theta_1<1} h\right)h$$

$$\frac{d\varphi}{dt}(b+\theta_1 h) = \frac{\partial f}{\partial y}(a+h,b+\theta_1 h) - \frac{\partial f}{\partial y}(a,b+\theta_1 h) = (Lagrange) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a+\theta_2 h, b+\theta_1 h)h$$

$$\Delta^2 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a+\theta_2 h, b+\theta_1 h)h^2 \Leftrightarrow \frac{\Delta^2}{h^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a+\theta_2 h, b+\theta_1 h)$$

מצד שני : $\Delta^2 = [f(a+h,b+h) - f(a+h)] - [f(a,b+h) - f(a,b)]$

$$\psi(t) \coloneqq f(t,b+h) - f(t,b)$$

$$\Delta^2 = \psi(a+h) - \psi(a) = \frac{\partial\psi}{\partial t}(a+\mu_1 h)h$$

$$\frac{\partial\psi}{\partial t}(a+\mu_1 h) = \frac{\partial f}{\partial x}(a+\mu_1 h, b+h) - \frac{\partial f}{\partial x}(a+\mu_1 h, b) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a+\mu_1 h, b+\mu_2 h)h$$
$$(0 < \mu_1, \mu_2 < 1)$$

$\lim_{h\to 0}\frac{\Delta^2}{h^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(a,b)$ לפי הרציפות של הנגזרות (ולכן של פונ' הרכבה).

בנוסף $\lim_{h\to 0}\frac{\Delta^2}{h^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(a,b)$ ולכן $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)$

מסקנה

אם קיימות $\frac{\partial}{\partial x_{j_r}}\left(\dots\frac{\partial f}{\partial x_{j_1}}\right)(x), \frac{\partial}{\partial x_{i_r}}\left(\dots\frac{\partial f}{\partial x_{i_1}}\right)(x)$ $x \in U$. והן רציפות בסביבה של a אזי הן שוות.

הגדרה

$$C^r(U) \coloneqq \{f: U \to \mathbb{R} : \text{כל נגזרות מסדר } r \text{ קיימות ורציפות}\}$$

$$D^r(U) \coloneqq \{f: U \to \mathbb{R} : \text{כל נגזרות מסדר } r \text{ קיימות}\}$$

תהי $f \in C^r(U)$, נגזרת מסדר r: $D_{i_1,\dots,i_r}f = \frac{\partial}{\partial x_{i_r}}\left(\frac{\partial}{\partial x_{i_{r-1}}}\left(\dots\frac{\partial f}{\partial x_{i_1}}\right)\right)$ (לדוגמא $\frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_1}\left(\frac{\partial}{x_2}\dots\frac{\partial f}{\partial x_3}\right)\right)$)

$$\mathrm{D}_{i_1,\dots,i_r} f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\left(\dots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\right) f \qquad \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} = \frac{\partial}{\partial x_j}\dots\frac{\partial}{\partial x_j}$$

$$D_{i_1,\dots,i_r} f = \frac{\partial^{\alpha_1+\dots+\alpha_n} f}{\partial x_1^{\alpha_1},\dots,\partial x_n^{\alpha_n}}$$

מולטי אינדקסים

$$\alpha = (\alpha_1,\dots,\alpha_n) \quad \alpha_j \in \{0\}\cup\mathbb{N}$$

n-מימד : $|\alpha| \coloneqq \alpha_1 + \dots \alpha_n = \left||\alpha|\right|_1$

$$\alpha! \coloneqq \alpha_1!\dots\alpha_n!$$

$$h\in\mathbb{R}^n \quad \alpha=(\alpha_1,\dots,\alpha_n) : h^\alpha \coloneqq h_1^{\alpha_1}\dots h_n^{\alpha_n}$$

הגדרה

$$\alpha = (\alpha_1,\dots,\alpha_n)$$

$$D^\alpha f \coloneqq \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1},\dots,\partial x_n^{\alpha_n}}$$

דיפרנציאל מסדר גבוה

יהי פולינום $p(x_1,\dots,x_n) = \sum c_\alpha x^\alpha = \sum\left(c_{\alpha_1,\dots,\alpha_n} x_1^{\alpha_1}\dots x_n^{\alpha_n}\right)$ כאשר $|\alpha| \le N$

לדוגמא $p(x,y,z) = 4x^2yz + 2z + x + y$

$$\deg(P) = \max\{|\alpha| : c_\alpha \neq 0\}$$

$$P(x) = \sum_{|\alpha|\le N} C_\alpha x^\alpha \qquad x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$$

הגדרה

P פולינום הומוגני ממעלה m אם $P(\lambda x) = \lambda^m P(x)$

לדוגמא $m = 3$ $\quad f(x,y,z) = x^3 + 2x^2y + 10z^3$

תרגיל

P הומוגני מסדר m $\Leftrightarrow P(x) = \sum_{|\alpha|=m} C_\alpha x^\alpha$

בינום של ניוטון מוכלל

$$(a+b)^r = \sum_{k=0}^{r}\binom{r}{k} a^k b^{r-k} = \sum_{\alpha_1+\alpha_2=r}\frac{r!}{\alpha_1!\,\alpha_2!} a^{\alpha_1} b^{\alpha_2}$$

בינום מוכלל

$$(\alpha_1+\dots\alpha_n)^r = \sum_{\alpha_1+\dots+\alpha_n=r}\frac{r!}{\alpha_1!\dots\alpha_n!} a_1^{\alpha_1}\dots a_n^{\alpha_n} = \sum_{|\alpha|=r}\frac{r!}{\alpha!} a^\alpha$$