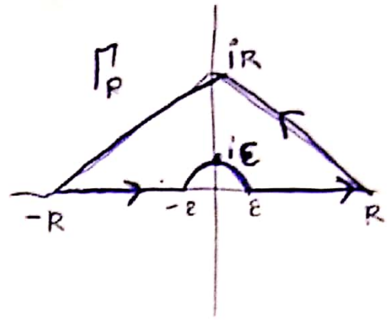
חשבו $\int_0^{\infty} \frac{\sin x}{x}\,dx$

$$\sin x = \operatorname{Im} e^{ix} \Rightarrow \frac{\sin x}{x} = \operatorname{Im}\left(\frac{e^{ix}}{x}\right)$$

יהי $0<\varepsilon<R$

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx = \int_{\varepsilon}^{R} \frac{e^{ix}-e^{-ix}}{x} = 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx$$

לפי משפט קושי:



$$\int_{\Gamma_R} \frac{e^{iz}}{z}\,dz = 0$$

$$\|$$

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx - \int_{-\varepsilon \curvearrowleft \varepsilon} \frac{e^{iz}}{z}\,dz + \int_{R \nwarrow iR} \frac{e^{iz}}{z}\,dz + \int_{iR \swarrow -R} \frac{e^{iz}}{z}\,dz$$

הפכנו כיוון.

$$\Rightarrow 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx = \int_{-\varepsilon \curvearrowleft \varepsilon} \frac{e^{iz}}{z}\,dz - \int_{\nwarrow} \frac{e^{iz}}{z}\,dz - \int_{\swarrow} \frac{e^{iz}}{z}\,dz$$

נשאיף: $\varepsilon \to 0$

נקבל משמאל:

$$\lim_{\varepsilon\to 0} 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}\,dx = 2i\int_{0}^{R} \frac{\sin x}{x}\,dx$$

נקבל מימין:

$$\frac{e^{iz}}{z} = \frac{1}{z} + \frac{e^{iz}-1}{z}$$ נשתמש בפיתוח

$$\int\limits_{\underset{-\varepsilon\quad\varepsilon}{\curvearrowleft}} \frac{e^{iz}}{z}\,dz = \int\limits_{\underset{-\varepsilon\quad\varepsilon}{\curvearrowleft}} \frac{dz}{z} + \int\limits_{\underset{-\varepsilon\quad\varepsilon}{\curvearrowleft}} \frac{e^{iz}-1}{z}\,dz \underset{\substack{z=\varepsilon e^{it}\\ dz=i\varepsilon e^{it}dt\\ 0\le t\le\pi}}{=} \int_0^{\pi} \frac{i\varepsilon e^{it}}{\varepsilon e^{it}}\,dt + \int\limits_{\underset{-\varepsilon\quad\varepsilon}{\curvearrowleft}} \frac{e^{iz}-1}{z}\,dz =$$

$$= \pi i + \int\limits_{\underset{-\varepsilon\quad\varepsilon}{\curvearrowleft}} \frac{e^{iz}-1}{z}\,dz .$$

נשים לב:

$$e^{iz} = 1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \dots \Rightarrow e^{iz}-1 = iz - \frac{z^2}{2} - \frac{iz^3}{6} + \dots$$

$$\Rightarrow \frac{e^{iz}-1}{z} = i - \frac{z}{2} - \frac{iz^2}{6} + \dots \xrightarrow[z\to 0]{} i$$

לכן, נגדיר:

$$g(z) = \begin{cases} \frac{e^{iz}-1}{z} & z\neq 0 \\ i & z=0 \end{cases}$$

g רציפה ב- $|z|\le 1$ לכן $|g|$ חסומה שם.

כלומר: קיים M כך שלכל $|z|\le 1$ $|g(z)|\le M$.

נשאיף $\varepsilon \to 0$:

$$0 \le \left|\int\limits_{\underset{-\varepsilon\quad\varepsilon}{\curvearrowleft}} \frac{e^{iz}-1}{z}\,dz\right| \underset{\text{משפט ההערכה.}}{\le} M\pi\varepsilon \xrightarrow[\varepsilon\to 0]{} 0$$

לכן:

$$\lim_{\varepsilon\to 0} \int\limits_{\underset{-\varepsilon\quad\varepsilon}{\curvearrowleft}} \frac{e^{iz}-1}{z}\,dz = 0$$

כעת:

$$2i\int_0^R \frac{\sin x}{x}\,dx = \pi i - \int\limits_{R}^{iR} \frac{e^{iz}}{z}\,dz - \int\limits_{iR}^{-R} \frac{e^{iz}}{z}\,dz$$

$$\left| \int_{R \to iR} \frac{e^{iz}}{z}\, dz \right| \underset{\uparrow}{=} \left| \int_0^R \frac{e^{-t} \cdot e^{(R-t)i}(-1+i)\, dt}{R-t+it} \right| \underset{\uparrow}{\leq}$$

$z = R - t + it$
$dz = (-1+i)\, dt$
$iz = -t + (R-t)i$
$e^{iz} = e^{-t} e^{(R-t)i}$

$0 \leq t \leq R$

(לפי פרמטריזציה של קו ישר:)

(לכל t) $e^{-t} > 0$

$|e^{(R-t)i}| = 1$

$|-1+i| = \sqrt{2}$

$$\leq \int_0^R \frac{e^{-t} \cdot \sqrt{2}\, dt}{|R-t+it|} \underset{\uparrow}{\leq} \int_0^R \frac{\sqrt{2}\, e^{-t} \sqrt{2}}{R}\, dt = \frac{2}{R} \int_0^R e^{-t}\, dt$$

$\min_{0 \leq t \leq R} |R-t+it| = \frac{R}{\sqrt{2}}$

$$= \frac{2}{R}(1 - e^{-R}) < \frac{2}{R}$$

$$0 \leq \left| \int_{R \to iR} \frac{e^{iz}}{z}\, dz \right| \leq \frac{2}{R} \xrightarrow[R \to \infty]{} 0$$ (קשוני)

לכן:
$$\lim_{R \to \infty} \int_{R \to iR} \frac{e^{iz}}{z}\, dz = 0$$

באופן דומה נקבל ,
$$\lim_{R \to \infty} \int_{iR \to -R} \frac{e^{iz}}{z}\, dz = 0$$

לכן מסקנה
$$2i \int_0^\infty \frac{\sin x}{x}\, dx = \pi i$$

$$\Rightarrow \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$