תרגיל 5 - פתרון.

שאלה 1:

$f(x)=x \quad , \ a=0, \ b=1.$

נחלק את הקטע $[0,1]$ ל-n קטעים שווים, כלומר

$$\Delta x_k = \frac{b-a}{n} = \frac{1}{n}$$

ונבחר את ξ_k להיות זהה ל-x_k. במקרה זה נקבל:

$$x_0=0, \ x_1=\frac{1}{n}, \ x_2=\frac{2}{n}, \ \dots, \ x_{n-1}=\frac{n-1}{n}, \ x_n=\frac{n}{n}=1$$

$$f(\xi_1)=\frac{1}{n}, \ f(\xi_2)=\frac{2}{n}, \ \dots, \ f(\xi_n)=\frac{n}{n} \quad \Rightarrow f(\xi_k)\cdot\Delta x_k=\frac{k}{n}\cdot\frac{1}{n}$$

$$\Rightarrow \int_0^1 x\,dx = \lim_{n\to\infty}\frac{1+2+\dots+n}{n^2} = \lim_{n\to\infty}\frac{n(n+1)}{2n^2} = \frac{1}{2}$$

שאלה 2:

א)

$$\int_1^3 x^3\sqrt{x^2-1}\,dx = \int_1^3 x^2\cdot x\sqrt{x^2-1}\,dx = \int_0^8 (1+t)\sqrt{t}\cdot\frac{1}{2}dt =$$

$$\left[\begin{array}{lll} x^2-1=t \ \text{הצבה} & & \text{שינוי גבולות:} \\ \Rightarrow x^2=1+t & 2x\,dx=dt & x=1 \Rightarrow 1=1+t \Rightarrow t=0 \\ x=\sqrt{1+t} & x\,dx=\frac{1}{2}dt & x=3 \Rightarrow 9-1=t \Rightarrow t=8 \end{array}\right]$$

$$= \frac{1}{2}\int_0^8\left(\sqrt{t}+t\sqrt{t}\right)dt = \frac{1}{2}\left.\frac{t^{3/2}}{3/2}\right|_0^8 + \frac{1}{2}\left.\frac{t^{5/2}}{5/2}\right|_0^8 = \frac{8^{3/2}}{3}+\frac{8^{5/2}}{5} =$$

$$= \frac{5\cdot 2^{9/2}+3\cdot 2^{15/2}}{15} = \frac{5\cdot 2^4\sqrt{2}+3\cdot 2^7\cdot\sqrt{2}}{15} = \frac{464\sqrt{2}}{15}$$

ב)

$$\int_0^1 \frac{x}{1+x^4}dx = \frac{1}{2}\int_0^1\frac{dt}{1+t^2} = \frac{1}{2}\left.\operatorname{arctg} t\right|_0^1 = \frac{1}{2}\left[\operatorname{arctg}1-\operatorname{arctg}0\right] =$$

$$\left[\begin{array}{lll} x^2=t & x=0 \Rightarrow t=0 & \text{הצבה:} \\ 2x\,dx=dt & x=1 \Rightarrow t=1 & \end{array}\right] \quad = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

ג)

$$\int_1^2 \frac{e^{1/x}}{x^2}dx = \int_1^{1/2} -e^t\,dt = \int_{1/2}^1 e^t\,dt = \left.e^t\right|_{1/2}^1 = e-\sqrt{e}$$

$$\left[\begin{array}{lll} \frac{1}{x}=t & \text{הצבה:} & x=\frac{1}{t} \\ -\frac{1}{x^2}dx=dt & & x=1 \Rightarrow t=1 \\ & & x=2 \Rightarrow t=\frac{1}{2} \end{array}\right]$$

(ד) $\displaystyle\int_0^{2\pi} \cos(5x)\cos x\,dx = \frac{1}{2}\int_0^{2\pi}\cos(4x)\,dx + \frac{1}{2}\int_0^{2\pi}\cos(6x)\,dx =$

$$\left[\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}\right]$$

$$= \frac{1}{8}\sin(4x)\Big|_0^{2\pi} + \frac{1}{12}\sin(6x)\Big|_0^{2\pi} = 0$$

(ה) $\displaystyle\int_0^{\pi/2} e^x\cos x\,dx = \frac{e^x(\cos x+\sin x)}{2}\Big|_0^{\pi/2} = \frac{e^{\pi/2}}{2} - \frac{1}{2} = \frac{1}{2}\left(e^{\pi/2}-1\right)$

[ראינו באחת התרגילים כי

$$\int e^x\cos x\,dx = \frac{e^x(\cos x+\sin x)}{2}$$]

שאלה 3: נוכיח כי

$$\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx = \begin{cases} 0 & m\neq n \\ \pi & m=n \end{cases}$$

הוכחה:

עבור $m\neq n$ נקבל:

$$\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\cos(m-n)x - \frac{1}{2}\int_{-\pi}^{\pi}\cos(m+n)x\,dx =$$

$$= \frac{1}{2}\sin(m-n)x\cdot\frac{1}{m-n}\Big|_{-\pi}^{\pi} - \frac{1}{2}\sin(m+n)x\cdot\frac{1}{m+n}\Big|_{-\pi}^{\pi} = 0$$

$m\neq n$

עבור $m=n$ נקבל:

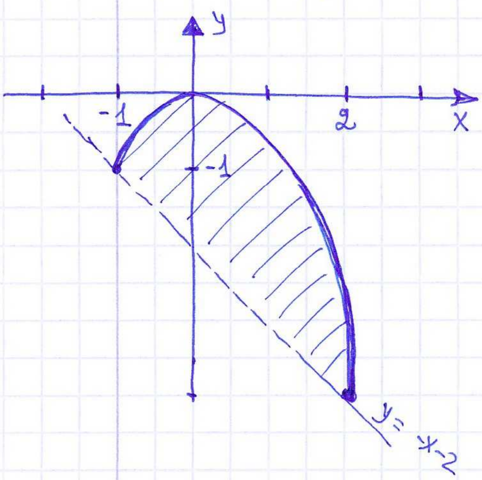
$$\int_{-\pi}^{\pi}\sin^2 mx\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1-\cos 2mx)\,dx = \frac{1}{2}x\Big|_{-\pi}^{\pi} - \frac{1}{4}\sin 2mx\Big|_{-\pi}^{\pi} =$$

$$= \frac{1}{2}\pi + \frac{1}{2}\pi = \pi$$

שאלה 4:

א

$$\begin{cases} y = -x^2 \\ x + y + 2 = 0 \end{cases} \Rightarrow \quad y = -x - 2$$



נק' חיתוך:

$$-x^2 = -x - 2$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

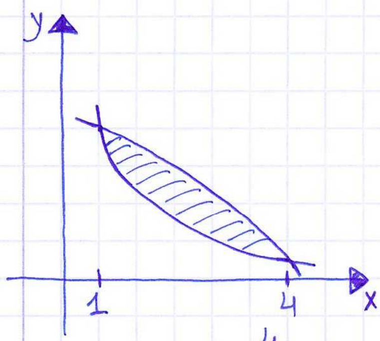
$$x_1 = -1 \qquad x_2 = 2$$

$$S = \int_{-1}^{2} \left[-x^2 - (-x-2)\right] dx = -\frac{x^3}{3} + \frac{x^2}{2} + 2x \Big|_{-1}^{2} =$$

$$= -\frac{8}{3} + 2 + 4 - \frac{1}{3} - \frac{1}{2} + 2 = 4\frac{1}{2}$$

ב

$$\begin{cases} y = 17 - x^2 \\ y = \dfrac{16}{x^2} \end{cases}$$



נק' חיתוך:

$$17 - x^2 = \frac{16}{x^2}$$

$$-x^4 + 17x^2 - 16 = 0 \qquad x^2 = t$$

$$t^2 - 17t + 16 = 0$$

$$t_1 = 16 \qquad t_2 = 1$$

$$x_{1,2} = \pm 4 \qquad x_{3,4} = \pm 1$$

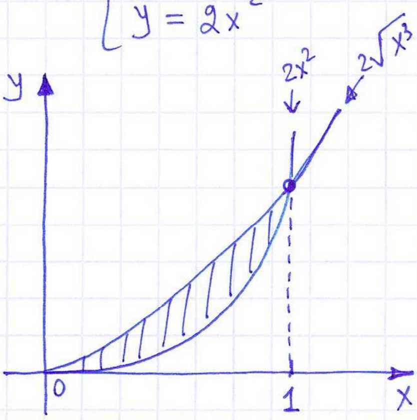
רביע ראשון! $\Rightarrow$ $x = 1,\ x = 2$

$$S = \int_{1}^{4} \left(17 - x^2 - \frac{16}{x^2}\right) dx = \left[17x - \frac{x^3}{3} + \frac{16}{x}\right]_1^4 =$$

$$= 17 \cdot 4 - \frac{4^3}{3} + 4 - 17 + \frac{1}{3} - 16 = 51 - 12 - 21 = 18$$

ג

$$\begin{cases} y^2 = 4x^3 \\ y = 2x^2 \end{cases} \qquad y = \pm\sqrt{4x^3}$$



$$S = \int_0^1 \left(2\, x^{3/2} - 2x^2\right) dx =$$

$$= 2\,\frac{x^{5/2}}{5/2} - \frac{2x^3}{3} \Big|_0^1 = \frac{4}{5} - \frac{2}{3} = \frac{12-10}{15} = \frac{2}{15}$$