Example questions, 2016

July 16, 2018

1. (a) The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{c}{r}. \tag{1}$$

(b) From E.L equation fro θ it follows that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0, \tag{2}$$

therefore $L \equiv L_\theta = mr^2\theta = const.$

(c) Substituting L in (1) it follows that

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{c}{r}. \tag{3}$$

(d) Assuming $r(0) = r_0, \dot{\theta}(0) = q, \dot{r}(0) = 0$ yields $E = mr_0^2q^2 - c/r_0$ and $L = mr_0^2q$.

(e) As the energy is conserved, we can get the distances by olving $U_{eff} = E$. This yields

$$r_{\min,\max} = \frac{c}{2|E|} \pm \sqrt{\frac{c^2}{4E^2} - \frac{L^2}{2m|E|}} \tag{4}$$

2. (a) Since the problem has cylindrical symmetry, it is conventeint to work with cylindrical frame. The Lagrangian is the given by (assuming the spring has a zero length at rest)

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}m\omega^2(r^2 + z_0^2), \tag{5}$$

(b) The Hamiltonian $\mathcal{H} = rp_r + \theta p_\theta - \mathcal{L}$ then reads

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}m\omega^2(r^2 + z_0^2). \tag{6}$$

(c) Hamilton's equations are

$$\begin{aligned}
\dot{r} &= \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{m}, && (7)\\
\dot{\theta} &= \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{mr^2}, && (8)\\
\dot{p}_r &= -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\theta^2}{mr^3} - m\omega^2 r, && (9)\\
\dot{p}_\theta &= -\frac{\partial \mathcal{H}}{\partial \theta} = 0. && (10)
\end{aligned}$$

From (10) it follows that the angular momentum p_θ is conserved.

(d) From (6) it follows that

$$U_{eff} = \frac{L_0^2}{2mr^2} + \frac{1}{2}m\omega^2(r^2 + z_0^2). \tag{11}$$

(e) Using $\Omega = \sqrt{U''_{eff}(r_0)/m}$ where r_0 solves $\dot{p}_r = 0$ (eq. (9)) gives $\Omega = \omega\sqrt{3}$.

(f) Performing similar calculation as in (1e) gives $r_{\text{min,max}} = \sqrt{a \pm \sqrt{a^2 - b}}$ where
$a = (2mE_0 - m^2\omega^2 z_0^2)/2m^2\omega^2 E_0, b = L_0^2/m^2\omega^2$.

3. (a) Schroedinger's equation for a free particle is

$$-\frac{\hbar^2}{2m}\psi'' = E\psi. \tag{12}$$

Denoting $k^2 = 2mE/\hbar^2$ the eigenvalues and eigenvectors are k^2 and

$$\psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}} \equiv |k\rangle, \tag{13}$$

respectively.

(b) There are no boundary conditions, therefore the eigenvalues are continuous.

(c) A straightforward calculation gives from $\langle\psi|\psi\rangle = 1$ $A = \sqrt{\alpha}$.

(d) Calculating $P(k) = |\langle k|\psi\rangle|^2$ where

$$\langle k|\psi\rangle = \sqrt{\alpha}\int_{-\infty}^{\infty} \frac{e^{-\alpha|x| - ikx}}{\sqrt{2\pi}}dx, \tag{14}$$

gives

$$P(k) = \frac{2\alpha^3}{\pi}\frac{1}{(\alpha^2 + k^2)^2}. \tag{15}$$

4. (a) Noticing that $|\psi|^2$ is a Gaussian we have that $\langle x \rangle = a$.

   (b) Similar considerations imply that since $\langle p_x \rangle = -i\hbar I$ where I is a real number, and since clearly $\langle p_x \rangle$ is a real, I must be zero and so $\langle p_x \rangle = 0$.

   (c) solving
$$\hat{p}|\psi_k\rangle = p|\psi_k\rangle, \tag{16}$$
for $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ and $|\psi_k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}}$ gives $p = \hbar k$. This means that (16) is an eigenvalues problem with eigenvectors and eigenvalues $|\psi_k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}}$ and $p = \hbar k$, respectively.

   (d) An integral expression for $|\langle \psi_k|\psi\rangle|^2$ for $k = p_0/\hbar$ is
$$|\langle \psi_k|\psi\rangle|^2 = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(x-2)^2}{2} - i\frac{p_0}{\hbar}x \right\} dx \right|^2. \tag{17}$$

5. Suppose the road has a deflection angle α with respect to the horizon. The rider "fills" the normal force N operated by the road, his weight mg and the centrifugal force $m\omega^2 R$ which balances the normal force in the horizontal directions. Solving
$$N\cos\alpha = mg \tag{18}$$
$$N\sin\alpha = m\omega^2 R \tag{19}$$
for α gives $\tan\alpha = \omega^2 R/g$.

6. (a) Denoting the orthonormal eigenstates of A corresponding to eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ by $|1\rangle$ and $|2\rangle$, respectively, it follows that
$$|\psi\rangle = |x\rangle = \langle 1|x\rangle|1\rangle + \langle 2|x\rangle|2\rangle. \tag{20}$$

   (b) The probabilities are given by
$$P(\lambda = 0) = |\langle 1|x\rangle|^2, \tag{21}$$
$$P(\lambda = 2) = |\langle 2|x\rangle|^2. \tag{22}$$

   (c) Decomposing A using the spectral theorem yields $A = 2|2\rangle\langle 2|$. Clearly
$$\langle A\rangle = 2P(\lambda = 2) = 2|\langle 2|x\rangle|^2. \tag{23}$$
On the other hand
$$\langle\psi|A|\psi\rangle = (\langle x|1\rangle\langle 1|+\langle x|2\rangle\langle 2|)2|2\rangle\langle 2|(\langle 1|x\rangle|1\rangle+\langle 2|x\rangle|2\rangle) = 2\langle x|2\rangle\langle 2|x\rangle = 2|\langle 2|x\rangle|^2. \tag{24}$$

   (d) From $U = e^{iAt}$ operating on $|\psi\rangle$ we have
$$|\psi(t)\rangle = e^{iAt}|x\rangle = \langle 1|x\rangle|1\rangle + \langle 2|x\rangle e^{2it}|2\rangle. \tag{25}$$

(e) Expressing $|y\rangle$ using the eigenstates of A gives

$$|y\rangle = \langle 1|y\rangle|1\rangle + \langle 2|y\rangle|2\rangle. \tag{26}$$

A mixed state occurs with some nonzero a probability $|\langle y|\psi(t)\rangle|^2 > 0$. Thus, the liftime of the state $|\psi(0)\rangle = |x\rangle$ is obtained by solving

$$|\langle y|\psi(t_0)\rangle|^2 = |a + e^{2it_0}b|^2 = |a|^2 + |b|^2 + 2\mathrm{Re}(abe^{2it_0}) = 0 \tag{27}$$

for t_0 where $a = \langle 1|y\rangle\langle 1|x\rangle$ and $b = \langle 2|y\rangle\langle 2|x\rangle$

7. (a) In x_1, y_1, x_2, y_2 system the Lagrangian reads

$$\mathcal{L} = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2). \tag{28}$$

Together with the constraint

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = L^2 \tag{29}$$

$\mathcal{L}$ can be written using 3 generalized coordinates.

(b) In CM and $\mathbf{r} = \mathbf{r_1} - \mathbf{r_2}$ ($r = |\mathbf{r_1} - \mathbf{r_2}|$) system the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2), \tag{30}$$

where $M = m_1 + m_2$ and $\mu = m_1m_2/M$ is the reduced mass. Adding the constraint $r = L$, (30) reduces to

$$\mathcal{L} = \frac{1}{2}M\dot{R}^2 + \frac{1}{2}\mu L^2\dot{\theta}^2. \tag{31}$$

(c) Since (31) does not depend on R it follows that $P_{CM} = M\dot{R}$ is conserved.

(d) Similarly, since (31) does not depend on θ it follows that the angular momentum $\mu L^2\dot{\theta}$ is conserved. Therefore $\dot{\theta}$ in (31) can be replaced by ω constant.