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1. (a) The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta})^2 - Ce^{-\alpha r}. \tag{1}$$

(b) E.L equation for r, θ are

$$m\ddot{r} = mr\dot{\theta}^2 + C\alpha e^{-\alpha r}, \tag{2}$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0, \tag{3}$$

(c) From (3) we deduce conservation of angular momentum.

(d) Taking an angular momentum L_0 the effective potential is given by

$$U_{eff} = \frac{L_0^2}{2mr^2} + Ce^{-\alpha r}. \tag{4}$$

(e) Using the initial conditions $r(0) = r_0, \dot{r}(0) = 0, \dot{\theta}(0) = \omega$ the initial energy is

$$E = \frac{1}{2}mr_0^2\omega^2 + Ce^{-\alpha r_0}. \tag{5}$$

Since the effective potential after a very long time is very small, we get from energy conservation

$$\dot{r} \approx \sqrt{r_0^2\omega^2 + \frac{2C}{m}e^{-\alpha r_0}} = Const, \tag{6}$$

and so $r \sim t$ and from conservation of angular momentum $\theta \sim t^{-2}$.

2. (a) The Hamiltonian is given by

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + U(r). \tag{7}$$

(b) Hamilton's equations read

$$\dot{r} = \frac{p_r}{m}, \tag{8}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}, \tag{9}$$

$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - U', \tag{10}$$

$$\dot{p}_\theta = 0. \tag{11}$$

(c) In Cartesian frame the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U(\sqrt{x^2 + y^2}). \tag{12}$$

Making the transformation $x \to x' = x + \epsilon y,\ y \to y' = y - \epsilon x$ result in a new Lagrangian

$$\mathcal{L}' = \frac{1}{2}m\left((\dot{x} + \epsilon\dot{y})^2 + (\dot{y} - \epsilon\dot{x})^2\right) - U(\sqrt{(x + \epsilon y)^2 + (y - \epsilon x)^2}). \tag{13}$$

The new Lagrangian (13) can be written as

$$\mathcal{L}' = \mathcal{L} + O(\epsilon^2), \tag{14}$$

so indeed the Lagrangian is invariant under infinitesimal rotation.

(d) Eq. (14) implies that Noether's theorem applies, and therefore

$$y\frac{\partial \mathcal{L}}{\partial \dot{x}} - x\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{x}y - m\dot{y}x = L, \tag{15}$$

which is the angular momentum in Cartesian frame, is conserved.

3. (a) Schroedinger equation is

$$-\frac{\hbar^2}{2m}\psi'' = E\psi. \tag{16}$$

The eigenstates solving the boundary problem (16) with the boundary conditions $\psi(0) = \psi(a) = 0$ are given by

$$\varphi_n = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi n}{a} x\right), \tag{17}$$

and the energies are $E_n = \frac{\pi\hbar^2}{2ma}n$.

(b) The normalized ψ is $\psi = 1/\sqrt{a}$.

(c) Writing

$$|\psi\rangle = \sum_n \langle\varphi_n|\psi\rangle|\varphi_n\rangle, \tag{18}$$

with

$$\langle \varphi_n | \psi \rangle = \frac{1}{a} \int_0^a \sin\left(\frac{\pi n}{a} x\right) dx = -\frac{2}{\pi(2n-1)} \tag{19}$$

we have that

$$\psi(x,t) = -\frac{2}{\pi\sqrt{a}} \sum_{n=1}^{\infty} \frac{e^{-iE_n t/\hbar}}{2n-1} \sin\left(\frac{\pi n}{a} x\right). \tag{20}$$

(d) The probability to find the particle at the nth level is

$$P_n = |\langle \varphi_n | \psi \rangle|^2 = \frac{4}{\pi^2 (2n-1)^2}. \tag{21}$$

4. (a) It is easily seen that

$$B|\varphi\rangle = 0. \tag{22}$$

(b) A straightforward calculation implies

$$C|\chi\rangle = |\mu\rangle, \tag{23}$$
$$C|\mu\rangle = |\chi\rangle. \tag{24}$$

(c) The two eigenstates of C are $|\varphi\rangle$ and $|\eta\rangle = (|\chi\rangle + |\mu\rangle)/\sqrt{2}$ with eigenvalues 0 and 1, respectively.

(d) Using the spectral theorem we obtain

$$C = |\eta\rangle\langle\eta| = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \tag{25}$$

(e) Expressing $|\psi\rangle$ in terms of $|\varphi\rangle, |\eta\rangle$ results in

$$|\psi\rangle = \frac{1}{2} \left((1+i)|\varphi\rangle + (1-i)|\eta\rangle \right), \tag{26}$$

which implies $P(\lambda_C = 0) = P(\lambda_C = 1) = 1/2$.

5. (a) Taking the weight mg to be equal to the centrifugal force $m\omega^2 R$ we obtain $\omega = \sqrt{g/R}$.

(b) $a_c = 2v\omega$ directed right (in a frame where the radial direction is pointing up).

(c) Let α be the deflection angle due to a_c, then

$$\tan\alpha = a_c/g = 2v/\sqrt{Rg}. \tag{27}$$