$$\boxed{(M+nm)\dot{R} = -m\sum_i \dot{r}_i'}$$

$$= -\frac{1}{2}(M+nm)\dot{R}^2 + \frac{1}{2}m\sum_i \dot{r}_i'^2 - U$$

$$L = \frac{1}{2}m\sum_i \dot{r}_i'^2 - \frac{1}{2}(M+nm)\left[-\frac{m}{M+nm}\sum_i \dot{r}_i'\right]^2 - U$$

$$L = \underbrace{\frac{1}{2}m\sum_i \dot{r}_i'^2 - \frac{1}{2}\frac{m^2}{M+nm}\left(\sum_i \dot{r}_i'\right)^2}_{\text{אנרגיה קינטית.}} - \underbrace{U}_{\text{פוטנציאלית}}$$

$$H = \sum_j p_j \dot{q}_j - L$$

$$L = \frac{1}{2}m\dot{q}^2 - U(q)$$

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \Rightarrow \dot{q} = \frac{p}{m}$$

$$H = p\dot{q} - L = \frac{p}{m}\cdot p - \frac{1}{2}m\left(\frac{p}{m}\right)^2 + U(q)$$

$$H = 2\left(\text{אנרגיה קינטית}\right) - \left[\left(\text{אנרגיה קינטית}\right) - \left(\text{אנרגיה פוטנציאלית}\right)\right]$$

$$H = \frac{1}{2}\frac{p^2}{m} + U(q)$$

$$E = \text{אנרגיה פוטנציאלית} + \text{אנרגיה קינטית.}$$

$$H = \frac{1}{2}m\sum_i \dot{r}_i'^2 - \frac{1}{2}\frac{m^2}{M+nm}\left(\sum_i \dot{r}_i'\right)^2 + U$$

$$\dot{r}_i' \to p_i' \qquad \frac{\partial L}{\partial \dot{q}_i} p_i \qquad \frac{\partial L}{\partial \dot{r}_i'} = p_i'$$

$$p_i' = \frac{\partial L}{\partial \dot{r}_i'} = \frac{1}{2}m\cdot 2\dot{r}_i' - \frac{1}{2}\frac{m^2}{M+nM}\cdot 2\cdot\sum_j \dot{r}_j'$$

$$p_i' = m\dot{r}_i' - \frac{m^2}{M+nM}\sum_j \dot{r}_j'$$

הרצאה 20:

המשך הפונקציה:

$$\sum_i p_i' = m\sum_i \dot{r}_i' - \frac{m^2}{M+nm}\sum_i\sum_j \dot{r}_i'$$

$$= m\sum_i \dot{r}_i' - \frac{m^2}{M+nm}n\sum_i \dot{r}_i' = \frac{m(M+nm) - m^2 n}{M+nm}\sum_i \dot{r}_i'$$

$$\sum_i p_i' = \frac{mM}{M+nm}\sum_i \dot{r}_i'$$

$$p_i' = m\dot{r}_i' - \frac{m^2}{M+nm}\left(\sum_i \dot{r}_i'\right)$$

$$\boxed{\sum_j \dot{r}_j' = \frac{M+nm}{mM}\sum_j p_j'}$$

$$p_i' = m\dot{r}_i' - \frac{m^2}{M+nm}\cdot\frac{M+nm}{mM}\sum_j p_j'$$

$$\boxed{\dot{r}_i' = \frac{1}{m}p_i' + \frac{1}{M}\sum_j p_j'}$$

$$H = \frac{m}{2}\sum_i \dot{r}_i'^2 - \frac{1}{2}\cdot\frac{m^2}{(M+nm)}\left(\sum_i \dot{r}_i'\right)^2 + U$$

$$H = \frac{m}{2}\sum_i\left(\frac{1}{m}p_i' + \frac{1}{M}\sum_j p_j'\right)^2 - \frac{1}{2}\frac{m^2}{(M+nm)}\left(\frac{M+nm}{mM}\sum_i p_i'\right)^2 + U$$

$$H = \frac{m}{2}\left[\frac{1}{m^2}\sum_i p_i'^2 + \frac{2}{mM}\sum_i p_i'\sum_j p_j' + \frac{1}{M^2}\sum_i\left(\sum_j p_j'\right)^2\right] - \frac{1}{2}\frac{m^2(M+nm)^2}{(M+nm)m^2M^2}\left(\sum_i p_i'\right)^2 + U$$

$$= \underline{\frac{1}{2m}\sum_i p_i'^2} + \frac{1}{M}\left(\sum_i p_i'\right)^2 + \frac{nm}{2M^2}\left(\sum_i p_i'\right)^2 - \frac{1}{2}\frac{M+nm}{M^2}\left(\sum_i p_i'\right)^2 + U$$

$$H = \frac{1}{2m}\sum_i p_i'^2 + \frac{2M+nm-M-nm}{2M^2}\left(\sum_i p_i'\right)^2 + U$$

$$H = \frac{1}{2m}\sum_i p_i'^2 + \frac{1}{2M}\left(\sum p_i'\right)^2 + U$$

$$MR + \sum_i m r_i = 0$$

$$r_i - R = r_i' \qquad r_i = r_i' + R$$

$$MR + \sum_i m(r_i' + R) = 0$$

$$MR + mnR + m\sum_i r_i' = 0$$

$$(M+nm)\dot{R} + m\sum_i \dot{r}_i' = 0 \qquad \sum_i p_i' \overset{?}{=} M\dot{R} + \frac{n\dot{R}}{m} \qquad -p + [\ ]$$

צריך לזכור שהדבר זה יהיה עם תנעים קנונים

ולכן אי אפשר להציב ישירות ולהשוות!!!

אי אפשר להציב ישירות $\frac{p^2}{m}$ ולהגיד זה אנרגיה קינטית בלתי

אפשרית בהחלט!!!

<u>כעת נוכיח את הקשר בין המילטוניאן לבין אנרגיה.</u>

<u>שימור אנרגיה והמילטוניאן:</u>

$$\cancel{\frac{dL}{dt} =} \qquad L(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_n)$$

$$\sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \sum_i \frac{\partial L}{\partial q_i}\cdot\frac{dq_i}{dt} = \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\dot{q}_i$$

$$\frac{dL}{dt} = \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right) = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i$$

$$0=\frac{d}{dt}\Big[\underbrace{\sum_i \frac{\partial L}{\partial \dot q_i}\dot q_i - L}\Big] \qquad H=E$$

המילטוניאן. אנרגיה של המערכת

$\frac{dH}{dt}=0$ וכש המילטוניאן לא תלוי מפורשות ב t אזי

אם המילטוניאן תלוי בt מפורשות t

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\sum_i \frac{\partial H}{\partial q_i}\cdot\frac{dq_i}{dt}+\sum \frac{\partial H}{\partial p_i}\frac{dp_i}{dt}$$

$$\frac{\partial H}{\partial q_i}=-\dot p_i \qquad \frac{\partial H}{\partial p_i}=\dot q_i \quad \Leftarrow$$

נזכיר:

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\sum(-\dot p_i)\dot q_i+\sum \dot q_i \dot p$$

$$\boxed{\frac{dH}{dt}=\frac{\partial H}{\partial t}}$$

איך מוצאים קבועי תנועה?

- סוגרי פואסון

$f(p,q,t) \leftarrow$ מה זה אומר שf הוא קבוע תנועה?

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial q}\cdot\frac{d}{dt}(q)+\frac{\partial f}{\partial p}\frac{d}{dt}(p)$$

$$f(p_1,\dots p_n,q_1,\dots,q_n,t)$$

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\sum_i\frac{\partial f}{\partial q_i}\dot q_i+\sum_i\frac{\partial f}{\partial p_i}\dot p_i$$

$$\cancel{\dot q_i=\frac{\partial H}{\partial t}+\sum_i\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p}} \quad \dot q_i=\frac{\partial H}{\partial p_i};\ \dot p_i=-\frac{\partial H}{\partial q_i}$$

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\underbrace{\sum_i\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}}_{\text{סוגרי פואסון}}$$

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\{H,f\}$$

$$\{H,f\}=\sum_i\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}$$

f קבוע תנועה (אם לא תלוי מפורשות בזמן)

$$\{H,f\}=0 \quad \Leftrightarrow$$

תכונות סוגרי פואסון:

באופן כללי מסמנים $\{f,g\}$

$$\{f,g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

נראה מספר תכונות של סוגרי פואסון:

I) $\{f,g\} = -\{g,f\}$

II) $\{f, const\} = 0$

III) $\{f_1+f_2, g\} = \{f_1, g\} + \{f_2, g\}$

IV) $$\{f_1 f_2, g\} = \sum_i \frac{\partial f_1 f_2}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f_1 f_2}{\partial q_i}\frac{\partial g}{\partial p_i}$$

$$= \sum_i \left(f_1 \frac{\partial f_2}{\partial p_i} + f_2 \frac{\partial f_1}{\partial p_i}\right)\frac{\partial g}{\partial q_i} - \left[f_1 \frac{\partial f_2}{\partial q_i} + f_2 \frac{\partial f_1}{\partial q_i}\right]\frac{\partial g}{\partial p_i}$$

$$= f_1\left(\sum_i \frac{\partial f_2}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f_2}{\partial q_i}\frac{\partial g}{\partial p_i}\right) + f_2\left[\sum_i \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f_i}{\partial q_i}\frac{\partial g}{\partial p_i}\right)\right]$$

$$= f_1\{f_2, g\} + f_2\{f_1, g\} = \{f_1 f_2, g\}$$

V) $$\{f, q_k\} = \sum \frac{\partial f}{\partial p_i}\frac{\partial q_k}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial q_k}{\partial p_i} = \sum_i \frac{\partial f}{\partial p_i}\delta_{ik} = \frac{\partial f}{\partial p_k}$$

$$\{f, q_k\} = \frac{\partial f}{\partial p_k}$$

VI) $$\{f, p_k\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial p_k}{\partial q_i} - \frac{\partial f}{\partial q_i}\cdot\frac{\partial p_k}{\partial p_i} = -\sum_k \frac{\partial f}{\partial q_i}\delta_{ik} = -\frac{\partial f}{\partial q_k}$$

VII) $\{q_i, q_k\} = 0$

VIII) $\{p_i, p_k\} = 0$

IX) $\{p_i, q_k\} = \delta_{ik}$

X) $$\frac{\partial}{\partial t}\{f,g\} = \frac{\partial}{\partial t}\left[\sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right]$$

$$= \sum_i \left[\frac{\partial}{\partial t}\left(\frac{\partial f}{\partial p_i}\right)\frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial p_i}\frac{\partial}{\partial t}\frac{\partial g}{\partial q_i} - \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial q_i}\right)\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial t}\frac{\partial g}{\partial p_i}\right]$$

$$= \sum_i \frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial t}\right)\frac{\partial g}{\partial q_i} - \frac{\partial}{\partial q_i}\left(\frac{\partial f}{\partial t}\right)\frac{\partial g}{\partial p_i} + \sum_i \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i}\left(\frac{\partial g}{\partial t}\right) - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\left(\frac{\partial g}{\partial t}\right)$$

Jacobi ~~שוויון~~

XI) $\{f, \{g,h\}\} + \{g, \{h,f\}\} + \{h, \{f,g\}\} = 0$

פואסון:

f קבוע תנועה ! g קבוע תנועה.

$\{f,g\} \longrightarrow$ קבוע

רוצים להוכיח.

$$\frac{d}{dt}\{f,g\} = \frac{\partial}{\partial t}\{f,g\} + \{H,\{f,g\}\}$$

$$\boxed{\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{H,A\}}$$

עם הצבה מפורשת
בכתיב.

$$\cancel{\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{H,A\}} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} - \{f,\{g,H\}\} - \{g,\{H,f\}\}$$

$$= \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{f, \frac{\partial g}{\partial t}\right\} + \{f,\{H,g\}\} + \{\{H,f\},g\}$$

$$= \left\{\frac{\partial f}{\partial t} + \{H,f\}, g\right\} + \left\{f, \frac{\partial g}{\partial t} + \{H,g\}\right\}$$

$$= \left\{\frac{df}{dt}, g\right\} + \left\{f, \frac{dg}{dt}\right\}$$

f ו g קבועי תנועה

$$\frac{d}{dt}\{f,g\} = 0+0=0$$

$$\{f,g\} = \text{const}$$