פתרון תרגיל 4

1. לפי אינטגרציה בחלקים

$$\int xe^{-x}\mathrm{d}x = -xe^{-x} + \int e^{-x}\mathrm{d}x = -xe^{-x} - e^{-x} + c$$

2. לפי נוסחא טריגונומטרית

$$\int e^{2x}\sin^2 x\mathrm{d}x = \int e^{2x}(\frac{1-\cos 2x}{2})\mathrm{d}x = \frac{1}{2}\int e^{2x}\mathrm{d}x - \frac{1}{2}\int e^{2x}\cos 2x\mathrm{d}x = \frac{1}{4}e^{2x} - \frac{1}{2}\int e^{2x}\cos 2x\mathrm{d}x$$

את האינטגרל שנשאר מוצאים בעזרת אינטגרציה בחלקים

$$\int e^{2x}\cos 2x\mathrm{d}x = \frac{1}{2}e^{2x}\cos 2x + \int e^{2x}\sin 2x\mathrm{d}x = \frac{1}{2}e^{2x}\cos 2x + \frac{1}{2}e^{2x}\sin 2x - \int e^{2x}\cos 2x\mathrm{d}x$$

ולכן

$$\int e^{2x}\cos 2x\mathrm{d}x = \frac{1}{4}e^{2x}\cos 2x + \frac{1}{4}e^{2x}\sin 2x$$

כלומר התשובה הסופית היא

$$\frac{1}{4}e^{2x} - \frac{1}{8}e^{2x}\cos 2x + \frac{1}{8}e^{2x}\sin 2x + c$$

.3

$$\int x^3\sqrt{9-x^2}\mathrm{d}x$$

נציב $t^2 = 9 - x^2$ כלומר

$$t\mathrm{d}t = -x\mathrm{d}x$$

ולכן האינטגרל הוא

$$\int x^3\sqrt{9-x^2}\mathrm{d}x = \int x^2\sqrt{9-x^2}x\mathrm{d}x = \int (9-t^2)t(-t\mathrm{d}t) = -\int 9t^2\mathrm{d}t + \int t^4\mathrm{d}t$$
$$= -3t^3 + \frac{1}{5}t^5 + c = -3(\sqrt{9-x^2})^3 + \frac{1}{5}(\sqrt{9-x^2})^5 + c$$

.4

$$\int \frac{e^x}{e^x+e^{\frac{x}{2}}}\,\boldsymbol{dx} = \int \frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}}+1}\,dx \underset{u=e^{\frac{x}{2}}}{=} \int \frac{2du}{u+1} = 2ln\left|e^{\frac{x}{2}}+1\right| + C$$

$$\int x^7\sqrt{5+3x^4}\,dx = \int x^4 x^3\sqrt{5+3x^4}\,dx = \frac{1}{12}\int x^4\cdot 12x^3\sqrt{5+3x^4}\,dx$$ 5

$$g' = x^3\sqrt{5+3x^4}\ ,\quad f := x^4$$

$$g := \frac{1}{18}(5+3x^4)^{3/2}\ ,\quad f' = 4x^3$$

ולכן נקבל:

$$\int x^7\sqrt{5+3x^4}\,dx = x^4\cdot\frac{1}{18}(5+3x^4)^{3/2} - \int\frac{1}{18}(5+3x^4)^{3/2}\cdot 4x^3\,dx$$

$$= \frac{1}{18}x^4(5+3x^4)^{3/2} - \frac{2}{9}\int x^3(5+3x^4)^{3/2}dx = \frac{1}{18}x^4(5+3x^4)^{3/2} - \frac{1}{135}(5+3x^4)^{5/2} + C$$

.6

$$\int\frac{x^3\cdot e^{x^2}}{(x^2+1)^2}dx = x^2\cdot e^{x^2}\frac{-1}{2(x^2+1)} + \int\frac{2xe^{x^2}(1+x^2)}{2(1+x^2)}dx =$$

$$= -\frac{x^2e^{x^2}}{2(x^2+1)} + \int x\cdot e^{x^2}dx = -\frac{x^2e^{x^2}}{2(x^2+1)} + \frac{1}{2}e^{x^2} + C$$

.7

$\sin 2x = 2\sin x\cos x$ ולכן $\sin^2 x\cos^2 x = \frac{1}{4}\sin^2 2x$.

$\cos 2x = 1-2\sin^2 x$ ולכן $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ ו $\sin^4 x = \frac{1}{4}(1-\cos 2x)^2$.

נכפיל את שתי המשוואות האחרונות לקבל $\int\sin^6 x\cos^2 x\,dx = \int\sin^4 x\sin^2 x\cos^2 x\,dx =$

$$= \int\frac{1}{4}(1-\cos 2x)^2\frac{1}{4}\sin^2 2x\,dx = \frac{1}{16}\left[\int\sin^2 2x\,dx - \int 2\cos 2x\sin^2 2x\,dx + \int\cos^2 2x\sin^2 2x\,dx\right].$$

נחשב כל אחד מן האינטגרלים:

$\cos 4x = \cos 2\cdot 2x = 1-2\sin^2 2x$ ולכן $\int\sin^2 2x\,dx = \frac{1}{2}\int(1-\cos 4x)dx = \frac{1}{2}x - \frac{1}{8}\sin 4x + C$

$\int 2\cos 2x \sin^2 2x dx$, נבצע הצבה $t=\sin 2x$ ולכן $dt = 2\cos 2x dx$ ונקבל

$$\int 2\cos 2x \sin^2 2x dx = \int t^2 dt = \frac{1}{3}t^3 + C = \frac{1}{3}\sin^3 2x + C$$

$\left(\frac{1}{2}\sin 4x\right)^2 = \cos^2 2x \sin^2 2x$ אבל $\cos 8x = 1 - 2\sin^2 4x$ ולכן

$$\int \cos^2 2x \sin^2 2x dx = \frac{1}{4}\int \sin^2 4x dx = \frac{1}{8}\int (1-\cos 8x) dx = \frac{1}{8}x - \frac{1}{64}\sin 8x + C$$

נציב את כל התוצאות האלה לקבל את התשובה הסופית.

.8

$$\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx = \frac{1}{4}\int \frac{\cos 2x}{\sin^2 2x} dx$$

נציב $t = \sin 2x$ ואז

$$dt = 2\cos 2x dx$$

כלומר

$$\frac{1}{4}\int \frac{\cos 2x}{\sin^2 2x} dx = \frac{1}{8}\int \frac{dt}{t^2} = -\frac{1}{8t} + C = -\frac{1}{8\sin 2x} + C$$

.9

$$\int \frac{1}{1+\sin x + \cos x} dx$$

נשתמש בהצבה אוניברסלית ונקבל

$$\int \frac{1}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{1+t^2}{1+t^2+2t+1-t^2} \frac{2}{1+t^2} dt$$
$$= \int \frac{2}{2+2t} dt = \ln(1+t) + C = \ln(\tan\frac{x}{2}+1) + C$$

.10

$$\int \frac{\sin 2x}{1+\sin^2 x} dx = \int \frac{2\sin x \cos x}{1+\sin^2 x} dx$$

נציב $t = \sin x$ ונקבל

$$\int \frac{2t}{1+t^2} dt = \ln(1+t^2) = \ln(1+\sin^2 x)$$

$$\int \frac{1}{x^2-a^2}\,dx = \begin{cases} -\frac{1}{x}+C & \text{: } a=0 \text{ אם} \\ \frac{1}{2a}\left(\ln\left|\frac{x-a}{x+a}\right|\right)+C & ⊛ \text{ :אחרת} \end{cases}$$ .11

$$⊛ \quad a\neq 0 : \quad \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$$

$$1 = A(x+a) + B(x-a)$$

$$\Rightarrow \quad A = \frac{1}{2a} \quad ; \quad B = -\frac{1}{2a}$$

$$\Rightarrow \quad \int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\left(\int\frac{dx}{x-a} - \int\frac{dx}{x+a}\right.$$

$$\int \frac{x^4}{x^4-1}\,dx = \int 1 + \frac{1}{(x-1)(x+1)(x^2+1)}\,dx$$ .12

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Dx+E}{x^2+1}$$

$$A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Dx+E)(x^2-1) = 1$$

נציב $\rightarrow x=-1 \Rightarrow -4B = 1 \Rightarrow B = -\frac{1}{4}$

$x = 1 \Rightarrow 4A = 1 \Rightarrow A = \frac{1}{4}$

השוואת מקדמים
עבור x^3 $\Rightarrow \quad A+B+D=0$

$\frac{1}{4} - \frac{1}{4} + D = 0 \Rightarrow D = 0$

$x = 0 \Rightarrow A - B - E = 1 \Rightarrow E = -\frac{1}{2}$

$$\Rightarrow \int \frac{x^4}{x^4+1}\,dx = x + \frac{1}{4}\int\frac{dx}{x-1} - \frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{x^2+1}$$

$$= x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\arctan x + C$$

$$\int \frac{x^3+3x^2+5x+7}{x^2+2}\,dx = \int (x+3)\,dx + \int \frac{3x+1}{x^2+2}\,dx =$$ .13

$$\begin{array}{l} \quad x+3 \\ x^3+3x^2+5x+7 \;\big\lfloor\; x^2+2 \\ -\;x^3 \qquad\quad +2x \\ \hline \quad -\;3x^2+3x+7 \\ \qquad 3x^2 \qquad +6 \\ \hline \qquad\qquad 3x+1 \end{array}$$

$$= \frac{x^2}{2} + 3x + \frac{3}{2}\int \frac{2x}{x^2+2}\,dx + \int \frac{dx}{x^2+2} =$$

$$= \frac{x^2}{2} + 3x + \ln(x^2+2) + \frac{1}{2}\arctan\frac{x}{\sqrt{2}} + C$$

$$\int \frac{x^5}{(x^3+1)(x^3+8)}\,dx = \begin{bmatrix} t = x^3+1 \\ dt = 3x\,dx \\ t+7 = x^3+8 \\ t-1 = x^3 \end{bmatrix} = \int \frac{(t-1)}{3t(t+7)}\,dt$$ .14

$$\frac{t-1}{t(t+7)} = \frac{A}{t} + \frac{B}{t+7}$$

$$t-1 = A(t+7) + Bt$$

33) $\Rightarrow$ $t=0 \Rightarrow -1 = 7A \Rightarrow A = -1/7$

$t=-7 \Rightarrow -8 = -7B \Rightarrow B = 8/7$

$$\Rightarrow \int \frac{x^5}{(x^3+1)(x^3+8)}\,dx = \frac{1}{3}\left(-\frac{1}{7}\right)\int \frac{dt}{t} + \frac{1}{3}\cdot\frac{8}{7}\int \frac{dt}{t+7} =$$

$$= -\frac{1}{21}\ln|t| + \frac{8}{21}\ln|t+7| + C$$

$$= -\frac{1}{21}\ln|x^3+1| + \frac{8}{21}\ln|x^3+8| + C$$

13) $\int \cos 3x \cos 2x \,dx = \int \frac{1}{2}(\cos 5x + \cos x)\,dx =$ .15

זהות:

$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$

$$= \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

.16

$$\int \cos x \cos 2x \cos 4x \; dx = \int \frac{1}{2}(\cos 3x + \cos x) \cos 4x \, dx =$$

$$= \int \frac{1}{4}(\cos 7x + \cos x + \cos 5x + \cos 3x) \, dx =$$

$$= \frac{1}{28} \sin 7x + \frac{1}{4} \sin x + \frac{1}{20} \sin 5x + \frac{1}{12} \sin 3x + C$$

.17

$$\int \tan^2 x \, dx = \int \frac{1}{\cos^2 x} - 1 \; dx = \tan x - x + C$$