תרגיל 1 – מתמטיקה לכימאים ג'

1. חשבו את הגבולות הבאים. פרטו את כל שלבי החישוב.

1.1. $$\lim_{n\to\infty}\frac{n^3-n^2+1}{3n^2-2n^3+2}=\lim_{n\to\infty}\frac{\frac{n^3-n^2+1}{n^3}}{\frac{3n^2-2n^3+2}{n^3}}=\lim_{n\to\infty}\frac{1-\frac{1}{n}+\frac{1}{n^3}}{\frac{3}{n}-2+\frac{2}{n^3}}=\frac{1-0+0}{0-2+0}=-0.5$$

1.2. $$\lim_{n\to\infty}\frac{n^2+5n+1}{2n-3n^3+4}=\lim_{n\to\infty}\frac{\frac{n^2}{n^3}+\frac{5n}{n^3}+\frac{1}{n^3}}{\frac{2n}{n^3}-\frac{3n^3}{n^3}+\frac{4}{n^3}}=\lim_{n\to\infty}\frac{\overbrace{\left(\frac{1}{n}\right)}^{\to 0}+\overbrace{\left(\frac{5}{n^2}\right)}^{\to 0}+\overbrace{\left(\frac{1}{n^3}\right)}^{\to 0}}{\underbrace{\left(\frac{2}{n^2}\right)}_{\to 0}-3+\underbrace{\left(\frac{4}{n^3}\right)}_{\to 0}}=\frac{0}{-3}=0$$

1.3.

$$\lim_{n\to\infty}\frac{\sqrt{4n^6+2n^8+111}+n}{\sqrt[4]{5n^2+n^3+13n^{16}+3}}=\lim_{n\to\infty}\frac{\frac{\sqrt{4n^6+2n^8+111}}{n^4}+\frac{n}{n^4}}{\frac{\sqrt[4]{5n^2+n^3+13n^{16}+3}}{n^4}}=\lim_{n\to\infty}\frac{\sqrt{\frac{4n^6}{n^8}+\frac{2n^8}{n^8}+\frac{111}{n^8}}+\frac{n}{n^4}}{\sqrt[4]{\frac{5n^2}{n^{16}}+\frac{n^3}{n^{16}}+\frac{13n^{16}}{n^{16}}+\frac{3}{n^{16}}}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{\underbrace{\left(\frac{4}{n^2}\right)}_{\to 0}+2+\underbrace{\left(\frac{111}{n^8}\right)}_{\to 0}}+\underbrace{\left(\frac{1}{n^3}\right)}_{\to 0}}{\sqrt[4]{\underbrace{\left(\frac{5}{n^{14}}\right)}_{\to 0}+\underbrace{\left(\frac{1}{n^{13}}\right)}_{\to 0}+13+\underbrace{\left(\frac{3}{n^{16}}\right)}_{\to 0}}}=\frac{\sqrt{2}}{\sqrt[4]{13}}=\sqrt[4]{\frac{4}{13}}$$

1.4. $$\lim_{n\to\infty}\frac{4n^3+n-6}{8n-n^2}=\lim_{n\to\infty}\frac{\frac{4n^3}{n^3}+\frac{n}{n^3}-\frac{6}{n^3}}{\frac{8n}{n^3}-\frac{n^2}{n^3}}=\lim_{n\to\infty}\frac{4+\overbrace{\left(\frac{1}{n^2}\right)}^{\to 0}-\overbrace{\left(\frac{6}{n^3}\right)}^{\to 0}}{\underbrace{\frac{8}{n^2}-\frac{1}{n}}_{\to 0^+}}\lim_{n\to\infty}\frac{4+\overbrace{\left(\frac{1}{n^2}\right)}^{\to 0}-\overbrace{\left(\frac{6}{n^3}\right)}^{\to 0}}{\underbrace{\underbrace{\left(\frac{1}{n}\right)}_{\to 0^+}\underbrace{\left(\frac{8}{n}-1\right)}_{\to -1}}_{0^-}}="\frac{4}{0^-}"=-\infty$$

1.5.

$$\lim_{n\to\infty}\frac{\sqrt{n^3+1}+n}{2n-1}=\lim_{n\to\infty}\frac{\frac{\sqrt{n^3+1}+n}{n^{\frac{3}{2}}}}{\frac{2n-1}{n^{\frac{3}{2}}}}=\lim_{n\to\infty}\frac{\sqrt{\frac{n^3+1}{n^3}}+\frac{n}{n^{\frac{3}{2}}}}{\frac{2n}{n^{\frac{3}{2}}}-\frac{1}{n^{\frac{3}{2}}}}=\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n^3}}+\frac{1}{n^{\frac{1}{2}}}}{\frac{2}{n^{\frac{1}{2}}}-\frac{1}{n^{\frac{3}{2}}}}=$$

$$\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n^3}}+\frac{1}{\sqrt{n}}}{\frac{2}{\sqrt{n}}-\frac{1}{n\sqrt{n}}}=\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n^3}}+\frac{1}{\sqrt{n}}}{\underbrace{\frac{1}{\sqrt{n}}}_{\to 0^+}\underbrace{\left(2-\frac{1}{n}\right)}_{\to 2}}="\frac{1}{0^+}"=\infty$$

2. חשבו את הגבולות הבאים, או הראו כי הם אינם קיימים:

$$\lim_{n\to\infty}\left(\sqrt{n^4+2n}-n^2\right)=\lim_{n\to\infty}\frac{\left(\sqrt{n^4+2n}-n^2\right)\left(\sqrt{n^4+2n}+n^2\right)}{\left(\sqrt{n^4+2n}+n^2\right)}=$$

2.1. $$\lim_{n\to\infty}\frac{\left(n^4+2n-n^4\right)}{\left(\sqrt{n^4+2n}+n^2\right)}=\lim_{n\to\infty}\frac{2n}{\left(\sqrt{n^4+2n}+n^2\right)}=\lim_{n\to\infty}\frac{\frac{2n}{n^2}}{\frac{\sqrt{n^4+2n}+n^2}{n^2}}$$

$$=\lim_{n\to\infty}\frac{\frac{2}{n}}{\sqrt{\frac{n^4}{n^4}+\frac{2n}{n^4}+\frac{n^2}{n^4}}}=\lim_{n\to\infty}\frac{\frac{2}{n}}{\sqrt{1+\frac{2}{n^3}+\frac{1}{n^2}}}=\frac{0}{\sqrt{1+0+0}}=0$$

2.2. $$\lim_{n\to\infty}\left(\sqrt{\underbrace{2^{\underbrace{\frac{n^3+1}{n-2}}_{\to\infty}}}_{\to\infty}}+\underbrace{\sin\underbrace{\left(\frac{1}{n}\right)}_{\to 0}}_{\to 0}\right)="\infty+0"=\infty$$

2.3. $$\lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n+\left(\frac{1}{3}\right)^{n+1}}{\left(\frac{1}{4}\right)^n-\left(\frac{1}{2}\right)^{n+2}}=\lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n+\frac{1}{3}\left(\frac{1}{3}\right)^n}{\left(\frac{1}{4}\right)^n-\frac{1}{4}\left(\frac{1}{2}\right)^n}=\lim_{n\to\infty}\frac{\frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n}+\frac{1}{3}\frac{\left(\frac{1}{3}\right)^n}{\left(\frac{1}{2}\right)^n}}{\frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^n}-\frac{1}{4}\frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n}}=\lim_{n\to\infty}\frac{1+\frac{1}{3}\left(\frac{2}{3}\right)^n}{\left(\frac{2}{4}\right)^n-\frac{1}{4}}$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{3}\overbrace{\left(\frac{2}{3}\right)^n}^{\to 0}}{\underbrace{\left(\frac{1}{2}\right)^n}_{\to 0}-\frac{1}{4}}=\frac{1}{-\frac{1}{4}}=-4$$

2.4. $$\lim_{n\to\infty}\frac{2^{2n}+3^{n+1}}{4^n-2^{n+2}}=\lim_{n\to\infty}\frac{4^n+3\cdot 3^n}{4^n-4\cdot 2^n}=\lim_{n\to\infty}\frac{\frac{4^n}{4^n}+3\cdot\frac{3^n}{4^n}}{\frac{4^n}{4^n}-4\cdot\frac{2^n}{4^n}}=\lim_{n\to\infty}\frac{1+3\cdot\overbrace{\left(\frac{3}{4}\right)^n}^{\to 0}}{1-4\cdot\underbrace{\left(\frac{1}{2}\right)^n}_{\to 0}}=\frac{1}{1}=1$$

2.5. $\lim_{n\to\infty}\left(3^n+(-3)^n\right)$

נשים לב:

$$3^n+(-3)^n=\begin{cases}3^n+3^n & n-even\\ 3^n-3^n & n-odd\end{cases}=\begin{cases}2\cdot 3^n & n-even\\ 0 & n-odd\end{cases}$$

תת הסדרה במקומות הזוגיים שואפת לאינסוף בעוד שתת הסדרה במקומות האי זוגיים היא אפס באופן תמידי (ולכן שואפת לאפס). לכן, לסדרה עצמה אין גבול. כלומר, הסדרה מתבדרת.

2.6. $$\lim_{n\to\infty}\underbrace{\left(\frac{4n+3}{5n-\sqrt{n}}\right)}_{\to\frac{4}{5}}^{n^2}="\left(\frac{4}{5}\right)^{\infty}"=0$$

2.7. $$\lim_{n\to\infty}\left(\frac{n+3}{n-2}\right)^n=\lim_{n\to\infty}\left(\frac{n-2+5}{n-2}\right)^n=\lim_{n\to\infty}\left(1+\frac{5}{n-2}\right)^n=\lim_{n\to\infty}\left(1+\frac{5}{n-2}\right)^{n\frac{n-2}{n-2}}=$$

$$\lim_{n\to\infty}\left(\underbrace{\left(1+\frac{5}{n-2}\right)^{n-2}}_{e^5}\right)^{\overbrace{\left(\frac{n}{n-2}\right)}^{\to 1}}=\left(e^5\right)^1=e^5$$

2.8.

$$\lim_{n\to\infty}\left(\frac{n^3-1}{n^3+4}\right)^{n^4}=\lim_{n\to\infty}\left(\frac{n^3+4-5}{n^3+4}\right)^{n^4}=\lim_{n\to\infty}\left(1+\frac{-5}{n^3+4}\right)^{n^4\frac{n^3+4}{n^3+4}}=$$

$$\lim_{n\to\infty}\left(\underbrace{\left(1+\frac{-5}{n^3+4}\right)^{n^3+4}}_{e^{-5}}\right)^{\overbrace{\frac{n^4}{n^3+4}}^{\to\infty}}="\left(\frac{1}{e^5}\right)^{\infty}"=0$$

2.9.

$$\lim_{n\to\infty}\left(1+\frac{n^2-1}{2n^4-2}\right)^{3n^2-5}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{2n^4-2}{n^2-1}}\right)^{3n^2-5}=\lim_{n\to\infty}\left(1+\frac{1}{\frac{2(n^4-1)}{n^2-1}}\right)^{3n^2-5}=$$

$$\lim_{n\to\infty}\left(1+\frac{1}{\frac{2(n^2-1)(n^2+1)}{n^2-1}}\right)^{3n^2-5}=\lim_{n\to\infty}\left(1+\frac{1}{2(n^2+1)}\right)^{3n^2-5}=\lim_{n\to\infty}\left(1+\frac{\frac{1}{2}}{(n^2+1)}\right)^{\frac{3n^2-5}{n^2+1}\cdot(n^2+1)}=$$

$$\lim_{n\to\infty}\left(\underbrace{\left(1+\frac{\frac{1}{2}}{(n^2+1)}\right)^{(n^2+1)}}_{e^{\frac{1}{2}}}\right)^{\overbrace{\frac{3n^2-5}{n^2+1}}^{\to 3}}=\left(e^{\frac{1}{2}}\right)^3=e^{\frac{3}{2}}$$

2.10. $\lim_{n\to\infty}\arctan n=\lim_{x\to\infty}\arctan x=\frac{\pi}{2}$

2.11. $$\lim_{n\to\infty}\frac{\sin\left(\frac{2}{n}\right)}{\frac{1}{n}}=\lim_{n\to\infty}\frac{2\sin\left(\frac{2}{n}\right)}{\frac{2}{n}}=\lim_{n\to\infty}2\underbrace{\left(\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}}\right)}_{\to 1}=2$$

2.12.

$$\lim_{n\to\infty}\frac{\sin\left(\frac{2}{n}\right)}{5\sin\left(\frac{5}{n}\right)}=\lim_{n\to\infty}\frac{\sin\left(\frac{2}{n}\right)}{5\sin\left(\frac{5}{n}\right)}\cdot\frac{\frac{2}{n}}{\frac{2}{n}}\cdot\frac{\frac{5}{n}}{\frac{5}{n}}=\lim_{n\to\infty}\left(\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}}\right)\cdot\left(\frac{\frac{5}{n}}{\sin\left(\frac{5}{n}\right)}\right)\cdot\frac{\frac{2}{n}}{5\cdot\frac{5}{n}}=$$

$$\lim_{n\to\infty}\underbrace{\left(\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}}\right)}_{\to 1}\cdot\underbrace{\left(\frac{\frac{5}{n}}{\sin\left(\frac{5}{n}\right)}\right)}_{\to 1}\cdot\frac{2}{25}=\frac{2}{25}$$

2.13.

$$\lim_{n\to\infty}\frac{\sqrt{1+\sin\left(\frac{3}{n}\right)}-1}{\sin\left(\frac{5}{n}\right)}=\lim_{n\to\infty}\frac{\sqrt{1+\sin\left(\frac{3}{n}\right)}-1}{\sin\left(\frac{5}{n}\right)}\cdot\frac{\sqrt{1+\sin\left(\frac{3}{n}\right)}+1}{\sqrt{1+\sin\left(\frac{3}{n}\right)}+1}=\lim_{n\to\infty}\frac{1+\sin\left(\frac{3}{n}\right)-1}{\sin\left(\frac{5}{n}\right)\sqrt{1+\sin\left(\frac{3}{n}\right)}+1}$$

$$=\lim_{n\to\infty}\frac{\sin\left(\frac{3}{n}\right)}{\sin\left(\frac{5}{n}\right)\sqrt{1+\sin\left(\frac{3}{n}\right)}+1}=\lim_{n\to\infty}\frac{\frac{5}{n}\cdot\frac{3}{n}\sin\left(\frac{3}{n}\right)}{\frac{5}{n}\frac{3}{n}\sin\left(\frac{5}{n}\right)\sqrt{1+\sin\left(\frac{3}{n}\right)}+1}=$$

$$\lim_{n\to\infty}\underbrace{\left(\frac{\sin\left(\frac{3}{n}\right)}{\frac{3}{n}}\right)}_{\to 1}\cdot\underbrace{\left(\frac{\frac{5}{n}}{\sin\left(\frac{5}{n}\right)}\right)}_{\to 1}\cdot\underbrace{\left(\frac{\frac{3}{n}}{\frac{5}{n}}\right)}_{=\frac{3}{5}}\cdot\underbrace{\left(\frac{1}{\sqrt{1+\sin\left(\frac{3}{n}\right)}+1}\right)}_{\to\frac{1}{2}}=\frac{3}{5}\cdot\frac{1}{2}=\frac{3}{10}$$

בהצלחה! ☺