תרגיל 11 אינפי 3 - פתרונות

שאלה 1

חשבו את האינטגרלים הבאים:

.1

$$\int_0^1\int_0^1 \frac{x^2}{1+y^2}dxdy = \frac{1}{3}\int_0^1 \left.\frac{x^3}{(1+y^2)}\right|_0^1 dy = \frac{1}{3}\int_0^1 \frac{1}{(1+y^2)}dy = \frac{1}{3}\arctan y|_0^1 = \frac{\pi}{6}$$

.2

$$\int_1^2\int_3^4 \frac{1}{(x+y)^2}dxdy = \int_1^2 -\left.\frac{1}{x+y}\right|_3^4 dy = \int_1^2 (-\frac{1}{4+y}+\frac{1}{3+y})dy =$$

$$-\ln(y+4)+\ln(y+3)|_1^2 = -\ln 6+\ln 5+\ln 5-\ln 4 = \ln\frac{25}{24}$$

.3

$$\int_0^1\int_x^1 \sin(y^2)dydx = \int_0^1\int_0^y \sin(y^2)dxdy = \int_0^1 \sin(y^2)x|_0^y dy = \int_0^1 y\sin y^2 dy \underset{\substack{t=y^2\\dt=2ydy}}{=}$$

$$= \int_0^1 \sin t dt = -\cos t|_0^1 = 1$$

.4

$$\int_0^\pi\int_0^1 x^{2n-1}\cos(x^n y)dxdy = \int_0^1\int_0^\pi x^{2n-1}\cos(x^n y)dydx = \int_0^1 x^{n-1}\sin(x^n y)|_0^\pi dx$$

$$= \int_0^1 x^{n-1}\sin(\pi x^n)dx \underset{\substack{t=x^n\\dt=nx^{n-1}dx}}{=} \frac{1}{n}\int_0^1 \sin\pi t dt = -\frac{1}{\pi n}\cos\pi t\Big|_0^1 = \frac{1}{\pi n}$$

.5

$$\iint_D \sin^7 x\cdot e^{\sqrt{y}}dxdy = \int_0^1\int_{y-1}^{1-y}\sin^7 x\cdot e^{\sqrt{y}}dxdy$$

נשים לב שהפונקציה

$$F(y)=\int_{y-1}^{1-y}\sin^7 x\cdot e^{\sqrt{y}}dx$$

שווה זהותית ל 0 כי לכל $y\in[0,1]$ הפונקציה $\sin^7 x\cdot e^{\sqrt{y}}$ היא אי זוגית על הקטע הסימטרי $[y-1,1-y]$. ולכן גם האינטגרל הכפול שווה 0.

.6

$$\int_0^4\int_{\sqrt{y}}^2\frac{1}{1+x^6}dxdy=\int_0^2\int_0^{x^2}\frac{1}{1+x^6}dydx=\int_0^2\frac{1}{1+x^6}y\Big|_0^{x^2}dx=$$

$$\int_0^2\frac{x^2}{1+x^6}dx\underset{\substack{t=x^3\\ dt=3x^2dx}}{=}\frac{1}{3}\int_0^8\frac{1}{1+t^2}dt=\frac{1}{3}\arctan t\Big|_0^8=\frac{1}{3}\arctan 8$$

.7

$$\iint\limits_D\frac{1}{x}dxdy=\int_1^2\int_{x^2}^{x^3}\frac{1}{x}dydx=\int_1^2\frac{y}{x}\Big|_{x^2}^{x^3}dx=\int_1^2(x^2-x)dx$$

$$=\frac{1}{3}x^3-\frac{1}{2}x^2\Big|_1^2=\frac{8}{3}-2-\frac{1}{3}+\frac{1}{2}=\frac{5}{6}$$

.8

$$\int_0^{\ln 2}\int_0^{\sqrt{z}}\int_0^{2x}e^{x^2}dydxdz=\int_0^{\ln 2}\int_0^{\sqrt{z}}e^{x^2}y\Big|_0^{2x}dxdz=\int_0^{\ln 2}\int_0^{\sqrt{z}}2xe^{x^2}dxdz\underset{\substack{t=x^2\\ dt=2xdx}}{=}$$

$$\int_0^{\ln 2}\int_0^{z}e^tdt=\int_0^{\ln 2}e^t\Big|_0^zdz=\int_0^{\ln 2}(e^z-1)dz=e^z-z\Big|_0^{\ln 2}=2-\ln 2-1=1-\ln 2$$

.9

$$\int_0^1\int_1^{e^z}\int_{\frac{1}{y}}^1\ln xdxdydz=\int_0^1\int_1^{e^z}(x\ln x-x)\Big|_{\frac{1}{y}}^1dydz=\int_0^1\int_1^{e^z}(-1+\frac{1}{y}\ln y+\frac{1}{y})dydz=$$

$$-\int_0^1\int_1^{e^z}dydz+\int_0^1\int_1^{e^z}\frac{1}{y}(\ln y+1)dydz$$

החלק הראשון:

$$-\int_0^1\int_1^{e^z} dydz = -\int_0^1 (e^z-1)dz = -e^z+z\Big|_0^1 = 2-e$$

החלק השני:

$$\int_0^1\int_1^{e^z} \frac{1}{y}(\ln y+1)dydz \underset{\substack{t=\ln y\\ dt=\frac{1}{y}dy}}{=} \int_0^1\int_0^z (t+1)dtdz = \int_0^1 (\frac{1}{2}t^2+t)\Big|_0^z dz =$$

$$\int_0^1 (\frac{1}{2}z^2+z)dz = \frac{1}{6}z^3+\frac{1}{2}z^2\Big|_0^1 = \frac{2}{3}$$

ביחד מתקבל

$$\frac{8}{3}-e$$

שאלה 2

.1

$$\int_0^1 \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)dxdy$$

.2

$$\int_0^2 \int_{\sqrt{\frac{y}{2}}}^{5-y^2} f(x,y)dydx$$

.3

$$\int_{-1}^1 \int_{-1}^{\sqrt[3]{y}} f(x,y)dxdy + \int_1^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f(x,y)dxdy$$