In solving a linear non-homogeneous differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

or in operator notation,

$$Ly = f(x),$$

the right hand (forcing) function $f(x)$ determines the method of solution. Fundamentally, the general solution of this differential equation is

$$y = y_p + y_h$$

where y_p is the **particular solution** to the original differential equation, that is,

$$Ly_p = f(x)$$

and y_h is the general solution to the **homogeneous equation**, meaning

$$Ly_h = 0.$$

By the **principle of superposition**, we have

$$L(y_p + y_h) = L(y_p) + L(y_h)$$
$$= f(x) + 0$$
$$= f(x)$$

It must be emphasized that we will always begin by finding the general solution of the homogeneous case $Ly = 0$.

Differential Equations and their Operator Form

Differential Equation	Characteristic Eqn	Linear Operator	General Solution
$y'' + 3y' - 10y = 0$	$r^2 + 3r - 10 = 0$ $(r+5)(r-2) = 0$	$L = D^2 + 3D - 10$ $= (D+5)(D-2)$	$y = c_1 e^{-5x} + c_2 e^{2x}$
$y'' - 6y' + 9y = 0$	$r^2 - 6r + 9 = 0$ $(r-3)^2 = 0$	$L = D^2 - 6D + 9$ $= (D-3)^2$	$y = c_1 e^{3x} + c_2 x e^{3x}$
$y'' + 16y = 0$	$r^2 + 16 = 0$	$L = D^2 + 16$	$y = c_1 \cos 4x + c_2 \sin 4x$
$y^{(4)} + 18y'' + 81y = 0$	$(r^2 + 9)^2 = 0$	$L = (D^2 + 9)^2$	$y = c_1 \cos 3x + c_2 \sin 3x + x(c_3 \cos 3x + c_4 \sin 3x)$
$y^{(4)} + y^{(3)} - 12y'' = 0$	$r^4 + r^3 - 12r^2 = 0$ $r^2(r+4)(r-3) = 0$	$L = D^2(D+4)(D-3)$	$y = c_1 + c_2 x + c_3 e^{-4x} + c_4 e^{3x}$

The table of linear operators and solutions gives us a hint as to how to determine the annihilator of a function. The functions that correspond to a factor of an operator are actually annihilated by that operator factor.

How do we determine the annihilator?

Closely examine the following table of functions and their annihilators. Notice that the annihilator of a linear combination of functions is the **product of annihilators**.

Table of Annihilators

f(x)	Annihilator
$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$	D^{n+1}
e^{kx}	$D - k$
$x^n e^{kx}$	$(D - k)^{n+1}$
$\cos bx$ or $\sin bx$	$D^2 + b^2$
$x^n \cos bx$ or $x^n \sin bx$	$(D^2 + b^2)^{n+1}$
$e^{ax} \cos bx$ or $e^{ax} \sin bx$	$(D-a)^2 + b^2 =$ $D^2 - 2aD + a^2 + b^2$
$x^n e^{ax} \cos bx$ or $x^n e^{ax} \sin bx$	$\left[(D-a)^2 + b^2 \right]^{n+1} =$ $\left[D^2 - 2aD + a^2 + b^2 \right]^{n+1}$
$3x^2 - 2x + 5 + 7e^{-2x}$	$D^3 (D + 2)$
$3x + 5 + 7xe^{-2x} - e^{4x}$	$D^2 (D + 2)^2 (D - 4)$

The Annihilator Method

We can use the **annihilator method** if f and all of its derivatives are a finite set of linearly independent functions. That is, f must be one of the following function types:
- Polynomial
- Sine or cosine
- Exponential (this includes hyperbolic sine and hyperbolic cosine)
- $x^n e^{kx}$, $x^n \cos(kx)$ or $x^n \sin(kx)$
- A linear combination of the above.

If $f(x)$ is of this form, we seek a differential **annihilator** of f, L_f, so that $L_f(f) = 0$. We apply L_f to both sides of the differential equation to obtain a new homogeneous equation

$$L_f L y = 0.$$

The Annihilator Method:
1. **Write the differential equation in factored operator form.**
2. **Solve the homogeneous case $Ly = 0$.**
3. **Apply the annihilator of $f(x)$ to both sides of the differential equation to obtain a new homogeneous differential equation.**
4. **Identify the basic form of the solution to the new differential equation.**
5. **Determine the specific coefficients for the particular solution.**

As a simple example, consider
$$3y'' - y' - 2y = 2x^2 + 6x - 4.$$

Let us note that we expect the particular solution to be a quadratic polynomial.

Solution

We **first rewrite the differential equation in operator form**
$$(3D^2 - D - 2)y = 2x^2 + 6x - 4$$
and factor (if possible):
$$(3D + 2)(D - 1)y = 2x^2 + 6x - 4.$$

We now **identify the general solution to the homogeneous case** $(3D + 2)(D - 1)y = 0$.
Since the characteristic equation is
$$(3r + 2)(r - 1) = 0,$$
the roots are $r = 1$ and $r = -\frac{2}{3}$ so the solution of the homogeneous equation is
$$y_h = c_1 e^{-\frac{2}{3}x} + c_2 e^x.$$

Now we **identify the annihilator of the right side of the non-homogeneous equation**:
$$D^3(x^2 + 5x - 7) = 0$$

We **apply the annihilator to both sides of the differential equation** to obtain a new homogeneous equation:
$$D^3(3D + 2)(D - 1)y = D^3(2x^2 + 6x - 4)$$
giving
$$D^3(3D + 2)(D - 1)y = 0$$

The next step is critical because we must **distinguish between the homogenous solution and the particular solution to the original non-homogeneous case.**

We know that the solution is (be careful of the subscripts)

$$y = y_h + y_p$$

$$= c_1 e^{-\frac{2}{3}x} + c_2 e^x + Ae^{0x} + Bxe^{0x} + Cx^2 e^{0x}$$

$$= \underbrace{c_1 e^{-\frac{2}{3}x} + c_2 e^x}_{y_h} + \underbrace{A + Bx + Cx^2}_{y_p}$$

We must substitute $y_p = A + Bx + Cx^2$ into the original differential equation to determine the specific coefficients A, B, and C ($y_p' = B + 2Cx$ and $y_p'' = 2C$):

$$3(2C) - (B + 2Cx) - 2(A + Bx + Cx^2) = 2x^2 + 6x - 4$$

Expanding and equating like terms gives

$$-2C = 2 \qquad\qquad (C = -1)$$

$$-2C - 2B = 6 \qquad\qquad (B = -2)$$

$$6C - B - 2A = -4$$

giving $A = 0$, $B = -2$, and $C = -1$.

Therefore, **the general solution to the original non-homogeneous equation** is

$$y = \underbrace{c_1 e^{-\frac{2}{3}x} + c_2 e^x}_{y_h} + \underbrace{(-2x - x^2)}_{y_p} \qquad\qquad \text{(parentheses added for readability)}$$

Now consider

$$y'' - 2y' - 15y = 5x - 8e^{3x}$$

Because the characteristic equation for the corresponding homogeneous equation is

$$r^2 - 2r - 15 = 0,$$

we can write the differential equation in operator form as

$$(D^2 - 2D - 15)y = 5x - 8e^{3x}$$

which factors as

$$(D - 5)(D + 3)y = 5x - 8e^{3x}.$$

The **characteristic roots** are $r = 5$ and $r = -3$ of the homogeneous equation

$$(D - 5)(D + 3)y = 0$$

correspond to the **general homogeneous solution**

$$y_c = c_1 e^{5x} + c_2 e^{-3x}.$$

The **annihilator** of the right-hand side $5x - 8e^{3x}$ is $D^2(D-3)$ because applying this operator yields

$$D^2(D-3)(5x - 8e^{3x}) = D^2(5 - 15x - 24e^{3x} + 24e^{3x})$$
$$= D^2(5 - 15x)$$
$$= 0 + 0$$
$$= 0$$

Therefore, we apply $D^2(D+3)$ to both sides of the original differential equation to obtain

$$D^2(D-3)(D-5)(D+3)y = D^2(D-3)(5x - 8e^{3x})$$
$$= 0$$

We now solve the homogeneous equation

$$D^2(D-3)(D-5)(D+3)y = 0.$$

Again, we must be careful to distinguish between the factors that correspond to the particular solution and the factors that correspond to the homogeneous solution.

We know that the solution is (be careful of the subscripts)

$$y = y_h + y_p$$
$$= c_1 e^{5x} + c_2 e^{-3x} + Ae^{0x} + Bxe^{0x} + Ce^{3x}$$
$$= \underbrace{c_1 e^{5x} + c_2 e^{-3x}}_{y_h} + \underbrace{A + Bx + Ce^{3x}}_{y_p}$$

We must substitute $y_p = A + Bx + Ce^{3x}$ into the original differential equation to determine the specific coefficients A, B, and C. (It is worth noting that Ce^{3x} will only correspond to the exponential term on the right side since it cannot contribute to the elimination of the other terms. Thus, we have

$$9Ce^{3x} - 2(B + 3Ce^{3x}) - 15(A + Bx + Ce^{3x}) = 5x - 8e^{3x}$$

Expanding and equating like terms yields

$$(9C - 6C - 15C)e^{3x} - 15Bx + (-2B - 15A) = 5x - 8e^{3x}$$

which results in the equations

$$-15B = 5$$
$$-2B - 15A = 0$$
$$-12C = 8$$

giving

$$A = \frac{2}{45}, \quad B = -\frac{1}{3}, \quad C = -\frac{2}{3}.$$

The general solution to the non-homogeneous equation is

$$y = \underbrace{c_1 e^{5x} + c_2 e^{-3x}}_{y_h} + \underbrace{\frac{2}{45} - \frac{1}{3}x - \frac{2}{3}e^{3x}}_{y_p}$$

<u>Special Case</u>: When solutions to the homogeneous case overlap with the particular solution

Let's modify the previous example a little to consider the case when the solutions to the homogeneous case overlap with the particular solution. Consider
$$y''-2y'-15y = 5x - 8e^{-3x}.$$
Because the characteristic equation for the corresponding homogeneous equation is
$$r^2 - 2r - 15 = 0,$$
we can write the differential equation in operator form as
$$(D^2 - 2D - 15)y = 5x - 8e^{-3x}$$
which factors as
$$(D-5)(D+3)y = 5x - 8e^{-3x}.$$

The **characteristic roots** $r = 5$ and $r = -3$ of the homogeneous equation
$$(D-5)(D+3)y = 0$$
correspond to the **general homogeneous solution**
$$y_c = c_1 e^{5x} + c_2 e^{-3x}.$$

Again, the **annihilator** of the right-hand side $5x - 8e^{-3x}$ is $D^2(D+3)$.

We apply $D^2(D+3)$ to both sides of the original differential equation to obtain
$$D^2(D+3)(D-5)(D+3)y = 0$$
or combining repeated factors,
$$D^2(D-5)(D+3)^2 y = 0.$$

Note that the particular solution y_p corresponds to the **repeated factor $D+3$** (since e^{-3x} appears in the homogeneous solution) **and the factor D^2**:
$$y_p = Ax + B + Cxe^{-3x}.$$

We must now substitute y_p and its derivatives into the original non-homogeneous differential equation to determine the coefficients A, B, and C. We have
$$y_p = Ax + B + Cxe^{-3x}$$
$$y'_p = A - 3Cxe^{-3x} + Ce^{-3x}$$
$$y'_p = 9Cxe^{-3x} - 6Ce^{-3x}$$
so substitution yields
$$9Cxe^{-3x} - 6Ce^{-3x} - 2(A - 3Cxe^{-3x} + Ce^{-3x}) - 15(Ax + B + Cxe^{-3x}) = 5x - 8e^{-3x}.$$

Combining like terms gives
$$-15Ax - (2A + 15B) + (9C + 6C - 15C)xe^{-3x} - (6C + 2C)e^{-3x} = 5x - 8e^{-3x}.$$

Equating the coefficients of like terms on each side gives

$$-15A = 5$$

$$2A + 15B = 0$$

$$8C = 8$$

so that $A = -1/3$, $B = -2/45$, and $C = 1$. Thus, the general solution to the differential equation is

$$y(x) = -\frac{1}{3}x - \frac{2}{45} + xe^{-3x} + c_1 e^{5x} + c_2 e^{-3x}.$$