THE DEFINITE INTEGRAL

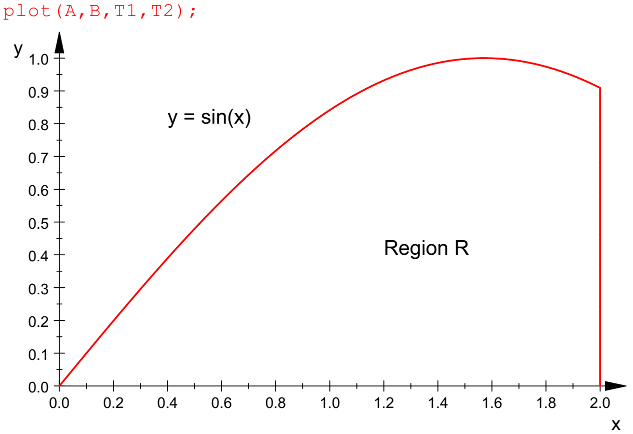
The integral $\int_a^b f(x)\,\mathrm{d}x$ is defined as $\lim_{n \to \infty} -\left(\sum_{k=0}^{n-1} f(c_k)\left(x_k - x_{k+1}\right)\right)$, where

$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ is a partition of $[a, b]$, $x_k < c_k < x_{k+1}$, and the limit is taken as $n \to \infty$ and the lengths of the subintervals determined by the partition go to 0. If $0 \le f(x)$ for all x in $[a, b]$, then the integral represents the area of the region bounded by the curve $y = f(x)$, the x -axis, and the lines $x = a$ and $x = b$. The sum $-\left(\sum_{k=0}^{n-1} f(c_k)\left(x_k - x_{k+1}\right)\right)$ represents the total area of the n rectangles with base $\left[x_k, x_{k+1}\right]$ and height $f(c_k)$, $k = 0 \ldots n-1$.

There are commands in **MuPAD** that allow us to draw the approximating rectangles and calculate the corresponding sums for partitions in which the x_k 's are evenly spaced and the c_k 's are either the left hand endpoints, the right endpoints, or the midpoints of the intervals $\left[x_k, x_{k+1}\right]$ determined by the partition.

As an example, we will let $f(x) = \sin(x)$, $a = 0$, and $b = 2$.

```
reset():
A:=plot::easy(sin(x),x=0..2,Colors=[RGB::Red]):
B:=plot::easy([2,x],x=0..sin(2),Colors=[RGB::Red]):
T1:=plot::Text2d("Region R",[1.2,0.4]):
T2:=plot::Text2d("y = sin(x)",[0.4,0.8]):
plot(A,B,T1,T2);
```



First we compute the exact value of the area of the region and it's numerical approximation.

```
J:=int(sin(x),x=0..2);float(J);
```

$1 - \cos(2)$

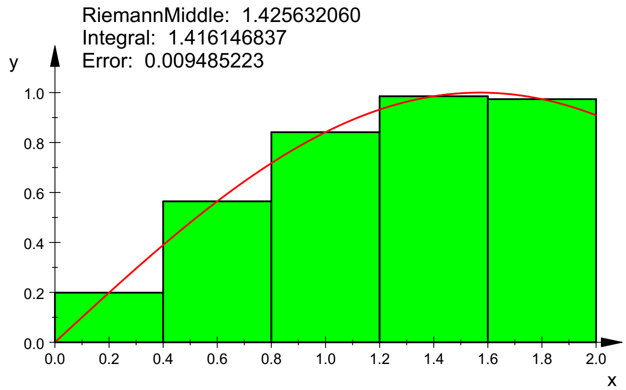
1.416146837

We will initially approximate the area of the region using 5 rectangles; the partition points will be equally spaced, and the c_k 's will be the midpoints of the subintervals; thus the partition points are 0, 0.4, 0.8, 1.2, 1.6 and 2, and the c_k 's are 0.2, 0.6, 1.0, 1.4 and 1.8.

```
C:=plot::Integral(plot::Function2d(sin(x),x=0..2),5,
IntMethod=RiemannMiddle,
```

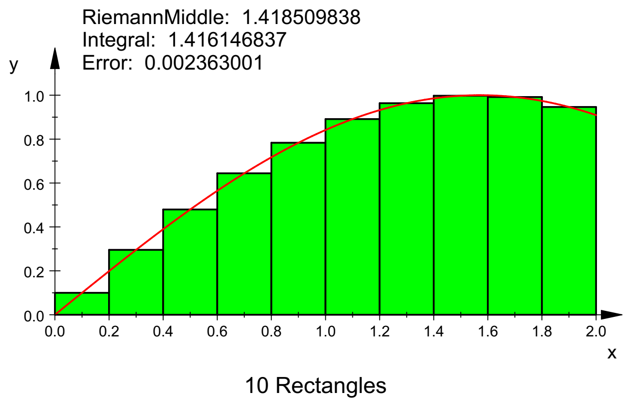
```
Color=RGB::Green,ShowInfo=[9,IntMethod,Integral,Error]):
```

```
plot(C,A);
```

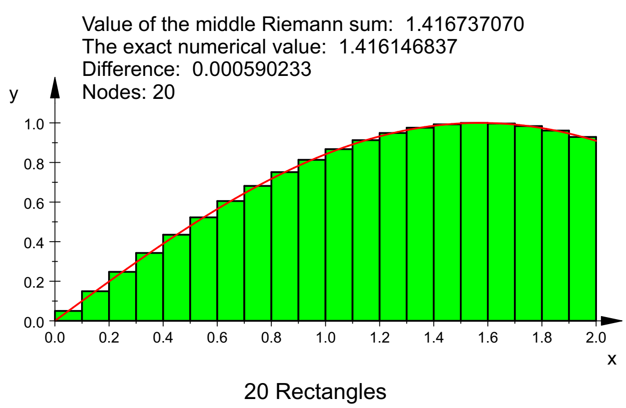


We now increase the number of rectangles.

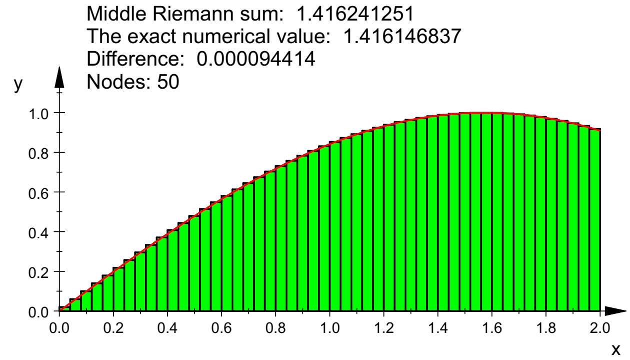
```
C:=plot::Integral(plot::Function2d(sin(x),x=0..2),10,
IntMethod=RiemannMiddle,
Color=RGB::Green,ShowInfo=[9,IntMethod,Integral,Error]):
plot(C,A,Footer="10 Rectangles");
```




```
C:=plot::Integral(plot::Function2d(sin(x),x=0..2),20,
IntMethod=RiemannMiddle,Color=RGB::Green,
ShowInfo = [9,IntMethod = "Value of the middle Riemann sum",
                   Integral = "The exact numerical value",
                   Error = "Difference",Nodes]):
plot(C,A,Footer="20 Rectangles");
```




```
C:=plot::Integral(plot::Function2d(sin(x),x=0..2),50,
IntMethod=RiemannMiddle,Color=RGB::Green,
ShowInfo = [9,IntMethod = "Middle Riemann sum",
                   Integral = "The exact numerical value",
                   Error = "Difference",Nodes]):
plot(C,A,Footer="50 Rectangles");
```



THE FUNDAMENTAL THEOREM OF CALCULUS

The definite integral of the function f (x) on the interval [a, b] is definite as

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum_{k=1}^{n} f(c_k)\left(x_k - x_{k-1}\right)$$

where $a = x_0 \le x_1 \le x_2 \le \ldots \le x_{n-1} \le x_n = b$ is a partition of [a, b] ,

$x_{k-1} \le c_k \le x_k$ for k = 1...n, and the limit is taken as $n \to \infty$ and the lengths

of the subintervals [x_{k-1} , x_k] determined by the partition go to 0.

The easiest and best way to evaluate the definite integral is by using the *Fundamental Theorem of Calculus*, which says that

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

where F is any antiderivative of f (i.e., F '(x) = f (x)).

EXAMPLE

Let $f(x) = x\cos(\pi x)$, $0 \le x \le 2\pi$.

We will use partitions in which the x_k 's are equally spaced and each c_k is the midpoint of [x_{k-1} , x_k]

First we illustrate the approximating rectangles using a partition with 20 points, followed by computing the limit of the Riemann sum.

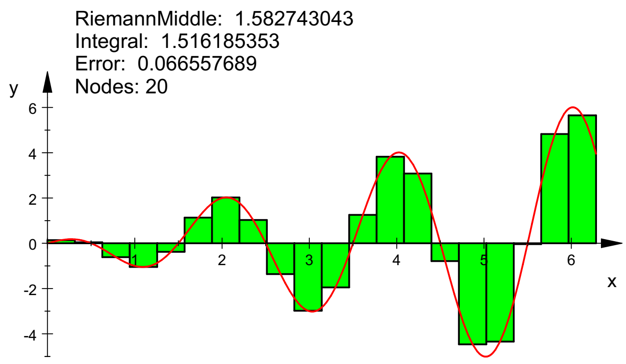
```
reset();
```

```
f:=x->x*cos(PI*x);
```

$x \to x\cos(\pi x)$

```
A:=plot::easy(f(x),x=0..2*PI,Colors=[RGB::Red]):
C:=plot::Integral(plot::Function2d(f(x),x=0..2*PI),20,
IntMethod=RiemannMiddle,Color=RGB::Green,
ShowInfo=[9,IntMethod,Integral,Error,Nodes]):
```

```
plot(C,A,Footer="20 rectangles with midpoints");
```



20 rectangles with midpoints

MuPAD thinks that the exact value of the integral is

```
J:=int(f(x),x=0..2*PI);float(J);
```

$$2\sin(2\pi^2) - \frac{2\sin(\pi^2)^2}{\pi^2}$$

1.516185353

To use the Fundamental Theorem of Calculus, we first use **MuPAD** to find an antiderivative F (x) for f (x). The command is

```
int(f(x),x);
```

$$\frac{\cos(\pi x) + \pi x \sin(\pi x)}{\pi^2}$$

(NOTE THAT **MuPAD** DOES NOT INCLUDE THE " + C".)

```
F:=x-> (cos(PI*x) + PI*x*sin(PI*x))/(PI^2);
```

$$x \to \frac{\cos(\pi x) + \pi x \sin(\pi x)}{\pi^2}$$

By the Fundamental Theorem the exact value of the integral is equal to F (2 π) - F (0) .

```
J;float(J);
```

$$2\sin(2\pi^2) - \frac{2\sin(\pi^2)^2}{\pi^2}$$

1.516185353

```
JFTC:=F(2*PI)-F(0);float(JFTC);
```

$$\frac{\cos(2\pi^2) + 2\pi^2\sin(2\pi^2)}{\pi^2} - \frac{1}{\pi^2}$$

1.516185353

The two results look different from the answer obtained above, but the two are actually equal. To see this we can attempt to use **MuPAD** to show that the expressions are identical as follows.

```
simplify(J-JFTC);
```

0

GEOMETRIC APPLICATIONS OF INTEGRATION

The definite integral can be used to solve a variety of problems from geometry; examples are finding areas between curves, lengths of curves, volumes and surface area of three dimensional solids, and centroids of plane regions.

AREA BETWEEN CURVES

If f (x) ≥ g (x) for all x in [a , b] then the area A of the region enclosed by the curves y = f (x), y = g (x), and the lines x = a a nd x = b is given by

$$A = \int_a^b (f(x) - g(x))\, dx .$$

In practice, a large part of the problem in using this formula involves determining exactly where f (x) ≥ g (x) and where g (x) ≥ f (x). We illustrate with an example:

EXAMPLE

Find the total area of all regions enclosed by the curves

f(x) = x + sin(2x) and g(x) = x^3 .

The first thing to do is to plot the graphs:

```
reset();
```

```
f:=x->x+sin(2*x); g:=x->x^3;
```

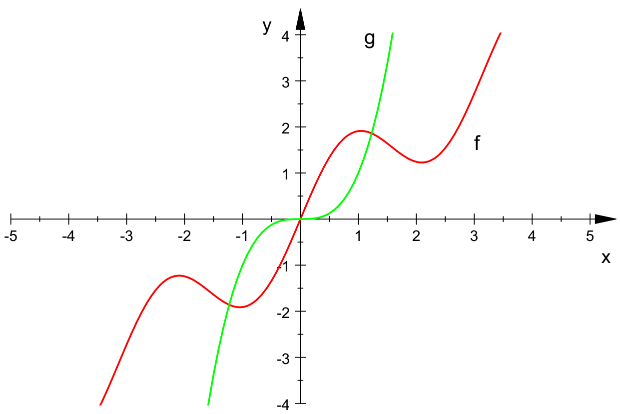
$x \to x + \sin(2\,x)$

$x \to x^3$

```
A:=plot::easy(f(x),x=-5..5,Colors=[RGB::Red]):
Tfcurve:=plot::Text2d("f",[3,1.5]):
```

```
B:=plot::easy(g(x),x=-5..5,Colors=[RGB::Green]):
Tgcurve:=plot::Text2d("g",[1.1,3.8]):
```

```
plot(A,B,Tfcurve,Tgcurve,ViewingBoxYRange=-4..4);
```



Note that the curves intersect at the origin, since f(0) = g (0) = 0. Next, we try find the X-coordinates of the other two points of intersection of the curves.

```
solve(f(x)=g(x),x);
```

$\text{solve}(x + \sin(2\,x) - x^3 = 0,\ x)$

Apparently there is no "nice" formula for the solutions, but we can still use "**fsolve**" to find decimal representations for them.

```
numeric::fsolve(f(x)=g(x),x=(-2)..(-0.5));
reset():f:=x->x+sin(2*x): g:=x->x^3:
```

$[x = -1.229835717]$

```
numeric::fsolve(f(x)=g(x),x=(.5)..(2));
reset():f:=x->x+sin(2*x): g:=x->x^3:
```

$[x = 1.229835717]$

```
X[1]:=-1.229835717;X[2]:=-X[1];
```

-1.229835717

1.229835717

Now we determine the interval or intervals where f(x) ≥ g(x), and the interval or intervals where g(x) ≥ f(x). From the graph, one can see that g (x) ≥ f(x) for x in [X_1 , 0] and that f (x) ≥ g (x) for x in [0 , X_2].

Finally, we set up the integrals and evaluate them:

```
A1:=int(g(x)-f(x),x=X[1]..0);
```
1.072518608

```
A2:=int(f(x)-g(x),x=0..X[2]);
```
1.072518608

```
CalculatedArea:=A1+A2;
```
2.145037216

VOLUMES OF SOLIDS OF REVOLUTION

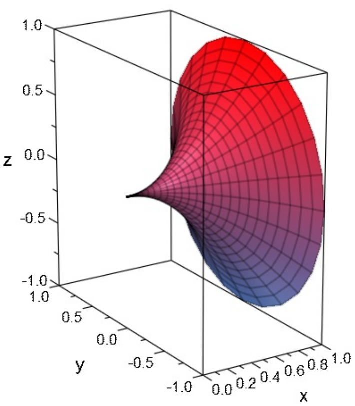
A solid generated by revolving a curve $y = f(x)$ about the X-axis is called a *solid of revolution*. Examples of such solids are cylinders, cones and spheres. The command **plot::XRotate(f(x),x=a..b);** plots the surface generated by revolving $y = f(x)$, $a \le x \le b$ about the X-axis.

As an example, let $f(x) = x^2$, $0 \le x \le 1$.

```
reset();
```

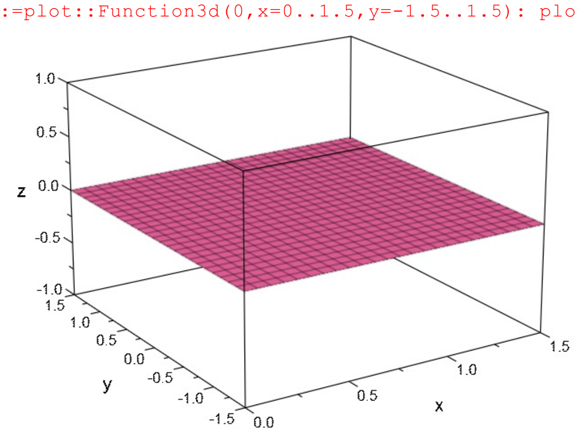
```
f:=x->x^2;
```
$x \to x^2$

```
P1:=plot::XRotate(f(x),x=0..1):
plot(P1);
```

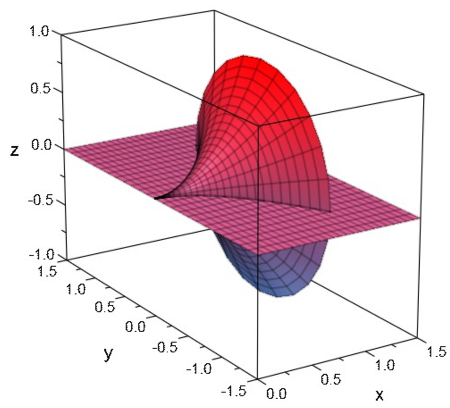


Click on the figure, and use the toolbar to rotate the solid, and change the appearance of the coordinate system.
You can also rotate the solid by moving the cursor while holding down the left mouse button.
You may plot several graphs on the same coordinate system.
For example, let's intersect the above graph with the X-Y plane.

```
P2:=plot::Function3d(0,x=0..1.5,y=-1.5..1.5): plot(P2);
```




```
plot(P1,P2);
```



The formula for the volume V of the solid resulting from rotating $y = f(x)$, $a \le x \le b$, about the X-axis is

$$V = \pi \int_a^b f(x)^2 \, dx$$

Thus, in the above example

$$V = \frac{\pi}{5},$$

and therefore

```
V:=PI*int(f(x)^2,x=0..1);float(V);
```

$$\frac{\pi}{5}$$

$$0.6283185307$$

Partial fractions

Example 1

```
23 + (x^4 + x^3)/(x^3 - 3*x + 2);
partfrac(23 + (x^4 + x^3)/(x^3 - 3*x + 2));
```

$$\frac{x^4 + x^3}{x^3 - 3x + 2} + 23$$

$$x + \frac{19}{9(x-1)} + \frac{2}{3(x-1)^2} + \frac{8}{9(x+2)} + 24$$

```
x^3/(x^2 + 3*I*x - 2);
partfrac(x^3/(x^2 + 3*I*x - 2))
```

$$\frac{x^3}{x^2 + 3x\,\mathrm{i} - 2}$$

$$x - \frac{7x + 6\,\mathrm{i}}{x^2 + 3x\,\mathrm{i} - 2} - 3\,\mathrm{i}$$

```
f := x^2/(x^2 - y^2);
partfrac(f, x), partfrac(f, y)
```

$$\frac{x^2}{x^2 - y^2}$$

$$\frac{y}{2(x-y)} - \frac{y}{2(x+y)} + 1,\ \frac{x}{2(x+y)} + \frac{x}{2(x-y)}$$

Example 2

The following example demonstrates the dependence of the partial fraction decomposition on the function factor:

```
partfrac(1/(x^2 - 2), x)
```

$$\frac{1}{x^2 - 2}$$

Note that the denominator does not factor over the rational numbers:

```
factor(x^2 - 2)
```

$$x^2 - 2$$

However, it factors over the extension containing . In the following calls, this extended coefficient field is implicitly assumed by factor and, consequently, by `partfrac`:

```
factor(sqrt(2)*x^2 - 2*sqrt(2));
partfrac(x/(sqrt(2)*x^2 - 2*sqrt(2)), x);
```

$$\sqrt{2}\,\left(x-\sqrt{2}\right)\,\left(x+\sqrt{2}\right)$$

$$\frac{\sqrt{2}\,x}{2\,\left(x^2-2\right)}$$

An extension of the coefficient field may also be enforced using the option `Adjoin`:

```
partfrac(1/(x^2 - 2), x, Adjoin = [sqrt(2)])
```

$$\frac{\sqrt{2}}{4\,\left(x-\sqrt{2}\right)}-\frac{\sqrt{2}}{4\,\left(x+\sqrt{2}\right)}$$

Example 3

Rational expressions of symbolic function calls may also be decomposed into partial fractions:

```
partfrac(1/(sin(x)^4 - sin(x)^2 + sin(x) - 1), sin(x))
```

$$\frac{1}{3\,\left(\sin(x)-1\right)}-\frac{\frac{\sin(x)^2}{3}+\frac{2\sin(x)}{3}+\frac{2}{3}}{\sin(x)^3+\sin(x)^2+1}$$

Example 4

The denominator can also be factored numerically over R_ or C_:

```
partfrac(1/(x^3 + 2), x, Domain = R_)
```

$$\frac{0.2099868416}{x+1.25992105}-\frac{0.2099868417\,x-0.529133684}{x^2-1.25992105\,x+1.587401052}$$

```
partfrac(1/(x^3 + 2), x, Domain = C_)
```

$$\frac{0.2099868416}{x+1.25992105}+\frac{-0.1049934208+0.1818539393\,i}{x-0.6299605249+1.091123636\,i}+\frac{-0.1049934208-0.1818539393\,i}{x-0.6299605249-1.091123636\,i}$$

Example 5

Use `Full` to factorize the denominator into linear factors symbolically:

```
partfrac(1/(x^3 + x - 2), x, Full)
```

$$\frac{1}{4\,(x-1)}+\frac{-\frac{1}{8}+\frac{3\sqrt{7}\,i}{56}}{x+\frac{1}{2}-\frac{\sqrt{7}\,i}{2}}-\frac{\frac{1}{8}+\frac{3\sqrt{7}\,i}{56}}{x+\frac{1}{2}+\frac{\sqrt{7}\,i}{2}}$$

Integration by parts and by change of variables

Typical applications for the rule of integration by parts

$$\int u'(x)\,v(x)\,\mathrm{d}\,x = u(x)\,v(x) - \int u(x)\,v'(x)\,\mathrm{d}\,x$$

are integrals of the form $\int p(x)\cos(x)\,\mathrm{d}\,x$ where $p(x)$ is polynomial. Thereby one has to use the rule in the way that the polynomial is differentiated. Thus one has to choose $u'(x) = \cos(x)$.

```
intlib::byparts(hold(int)((x-1)*cos(x),x),cos(x)); // hold(object) prevents the evaluation of object.
intlib::byparts(hold(int)((x-1)*cos(x),x),x-1);
```

$$\sin(x)\,(x-1)-\int \sin(x)\,\mathrm{d}\,x$$

$$-\int \sin(x)\left(x-\frac{x^2}{2}\right)\mathrm{d}\,x-\cos(x)\left(x-\frac{x^2}{2}\right)$$

In particular with the ansatz $u'(x) = 1$ it is possible to compute a lot of the well-known standard integrals, like e.g. $\int \arcsin(x)\,\mathrm{d}\,x$.

```
intlib::byparts(hold(int)(arcsin(x),x),1)
```

$$x\arcsin(x)-\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}\,x$$

In order to determine the remaining integral one may use the method change of variable

$$\int f(g(x))\,g'(x)\,\mathrm{d}\,x = F(g(x))+c$$

with $g(x) = 1 - x^2$.

```
F:=intlib::changevar(hold(int)(x/sqrt(1-x^2),x), t=1-x^2)
```

$$\int \left(-\frac{1}{2\sqrt{t}}\right) \mathrm{d}\,t$$

Via backsubstition into the solved integral `F` one gets the requested result.

```
hold(int)(arcsin(x),x) = x*arcsin(x)-subs(eval(F),t=1-x^2)
```

$$\int \arcsin(x)\ \mathrm{d}\,x = x\arcsin(x) + \sqrt{1-x^2}$$

Applying change of variable with the integrator is problematic, since it may occur that the integrator will never terminate. For that reason this rule is used within the integrator only on certain secure places. On the other hand, this may also lead to the fact that some integrals cannot be solved directly.

```
f:= sqrt(x)*sqrt(1+sqrt(x));
int(f,x)
```

$$\sqrt{x}\,\sqrt{\sqrt{x}+1}$$

$$\int \sqrt{x}\,\sqrt{\sqrt{x}+1}\ \mathrm{d}\,x$$

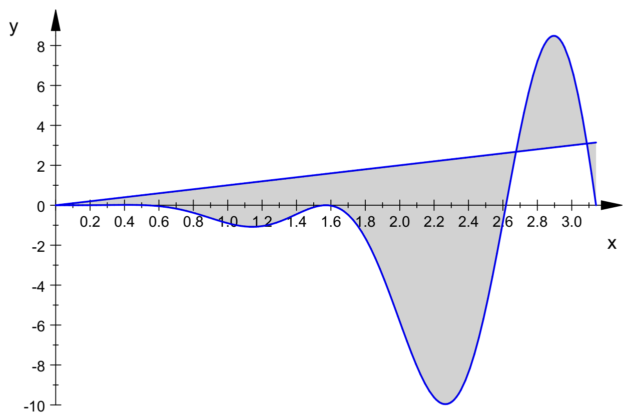
```
eval(intlib::changevar(hold(int)(f,x),t=sqrt(x))) | t=sqrt(x)
```

$$-\frac{4\,(\sqrt{x}+1)^{3/2}\,(42\sqrt{x} - 15\,(\sqrt{x}+1)^2 + 7)}{105}$$

Integration

Calculate the area between y=x and y=x^3*sin(2x)*cos(3x) in the interval [0,PI]

```
y1:=plot::Function2d(x, x = 0..PI):
y2:=plot::Function2d(x^3*sin(2*x)*cos(3*x), x = 0..PI):
h1 := plot::Hatch(y1, y2,   FillColor = RGB::Black,    FillPattern = DiagonalLines ):
plot(y1,y2,h1);
```




```
a1:=numeric::fsolve(x=x^3*sin(2*x)*cos(3*x),x=0)[1][2];
a2:=numeric::fsolve(x=x^3*sin(2*x)*cos(3*x),x=2.5)[1][2];
a3:=numeric::fsolve(x=x^3*sin(2*x)*cos(3*x),x=3.1)[1][2];
```

0.0

2.676321839

3.088396847

```
s1:=int(x-x^3*sin(2*x)*cos(3*x),x=a1..a2);
s2:=int(x^3*sin(2*x)*cos(3*x)-x,x=a2..a3);
s:=s1+s2;
```

9.407406281

1.500984532

10.90839081

Integrate $\int_0^2 x^x\ \mathrm{d}\,x$

```
numeric::int(x^x,x=0..3);
```

14.50853565

Integrate

Integrate $\int_{1}^{\infty}\int_{0}^{1}\frac{x}{y^2}\,\mathrm{d}x\,\mathrm{d}y$

```
int(int(x/y^2,x=0..1),y=1..infinity);
```

$$\frac{1}{2}$$

Integrate $\int_{-1}^{1}\int_{-1}^{1}\mathrm{e}^{x^2+x\,y+y^2}\,\mathrm{d}x\,\mathrm{d}y$

```
int(int(exp(x*y+x^2+y^2),x=-1..1),y=-1..1);
```

$$\int_{-1}^{1}\frac{\sqrt{\pi}\,\mathrm{e}^{\frac{3\,y^2}{4}}\left(\mathrm{erf}\left(\frac{y\,\mathrm{i}}{2}-\mathrm{i}\right)\mathrm{i}-\mathrm{erf}\left(\frac{y\,\mathrm{i}}{2}+\mathrm{i}\right)\mathrm{i}\right)}{2}\,\mathrm{d}y$$

```
numeric::int(int(exp(x*y+x^2+y^2),x=-1..1),y=-1..1)
```

$$9.375303465$$

Integrate $\int_{-2}^{3}\int_{x^2}^{x+6}x\,\mathrm{d}y\,\mathrm{d}x$

```
int(int(x,y=x^2..x+6),x=-2..3);
```

$$\frac{125}{12}$$

Integrate $\int_{2}^{4}\int_{-y}^{-\frac{1}{y^2}}(x^2\,y+x\,y^2)\,\mathrm{d}x\,\mathrm{d}y$

```
int(int(x^2*y+x*y^2,x=-y..-1/y^2),y=2..4);
```

$$-\frac{506059}{15360}$$