בס"ד

פתרון תרגיל 4

1

$f(x,y) = y\sin x$

$f_x = y\cos x = 0 \Rightarrow y=0 \,,\, \cos x = 0$

$f_y = \sin x = 0 \Rightarrow \sin x = 0$

$\Big\} \Rightarrow \begin{matrix} y=0 \\ \sin x = 0 \end{matrix}$ (לא יתכן $\cos x=0$, $\sin x=0$)

לכן: $x = \pi k$, $y=0$

קיבלנו את הנקודה: $(\pi k, 0, 0)$

$f_{xx} = -y\sin x = 0$

$f_{yy} = 0$

$f_{xy} = \cos x = \cos(\pi k) = 1$

$\Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1$

נקודת אוכף

2

① $f(x,y) = x^2 - e^{y^2}$

$f_x = 2x \qquad f_y = -2ye^{y^2}$

$f_{xx} = 2 \qquad f_{xy} = 0 \qquad f_{yy} = -2e^{y^2} - 4y^2e^{y^2}$

$f_x = 0 \Rightarrow x = 0$

$f_y = 0 \Rightarrow y = 0$

$\Big\} \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 2\cdot(-2) - 0 = -4 < 0$

לכן $(0,0,-1)$ נקודת אוכף.

② $k(x,y) = e^x \sin y$

$f_x = e^x\sin y = 0 \Rightarrow \sin y = 0$

$f_y = e^x\cos y = 0 \Rightarrow \cos y = 0$

$\Big\} \Rightarrow$ אין פתרון ולכן אין נקודות קריטיות

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3 $f(x,y) = e^{xy}$

$f_x = ye^{xy} = 0 \Rightarrow y = 0$

$f_y = xe^{xy} = 0 \Rightarrow x = 0$

$\Big\} \Rightarrow (0,0,1)$

נקודה קריטית

$f_{xx} = y^2 e^{xy} = 0$

$f_{yy} = x^2 e^{xy} = 0$

$f_{xy} = e^{xy} + xy\, e^{xy} = 1$

$\Rightarrow f_{xx} f_{yy} - f_{xy}^2 = -1$

אוכף

3

$f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

בתחום:

$\frac{1}{2}x^2 + y^2 \le 1 \Rightarrow g(x,y) = \frac{1}{2}x^2 + y^2$

על התחום:

$\frac{1}{2}x^2 + y^2 = 1$

$\nabla f = \lambda \nabla g$

$$\begin{cases} x = \lambda x \\ y = \lambda 2y \end{cases} \Rightarrow \lambda = 0 \longrightarrow x = 0,\ y = 0 \longrightarrow (0,0)$$

לא על התחום

$\lambda \neq 0 \longrightarrow$

$\lambda \neq \frac{1}{2} \longrightarrow y = 0$, x כלשהו $\longrightarrow \frac{1}{2}x^2 = 1 \longrightarrow x = \pm\sqrt{2}$

$(\pm\sqrt{2}, 0)$

$\lambda = \frac{1}{2} \longrightarrow x = 0$, y כלשהו $\longrightarrow y^2 = 1 \longrightarrow y = \pm 1 \longrightarrow$

$(0, \pm 1)$

בתוך התחום:

$f_x = x = 0 \quad f_y = y = 0 \longrightarrow (0,0)$

נקודה	$(0,0)$	$(\sqrt{2},0)$	$(-\sqrt{2},0)$	$(0,1)$	$(0,-1)$
$f(x,y)$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$

מינימום מוחלט:

$0 \leftarrow (0,0)$

מקסימום מוחלט: $1 \Leftarrow (\pm\sqrt{2}, 0)$

④

$f(x,y,z)=x+z \qquad x^2+y^2+z^2=1 \rightarrow g(x,y,z)=x^2+y^2+z^2$

$\nabla f=\lambda\nabla g \rightarrow \begin{cases} 1=\lambda\cdot 2x \\ 0=\lambda\cdot 2y \\ 1=\lambda\cdot 2z \end{cases} \rightarrow \begin{cases} \lambda=0 \rightarrow (0,0,0) \\ \lambda\neq 0 \rightarrow \begin{cases} y=0 \\ x=z \end{cases} \rightarrow \end{cases}$

לא מקיים את האילוץ

$2x^2=1$

$x=\pm\frac{1}{\sqrt{2}}$

נקודה	$\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$	$\left(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$
$f(x,y,z)$	$\sqrt{2}$	$-\sqrt{2}$

מקסימום: $f\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)=\sqrt{2}$

⑤

① $f(x,y)=x^2-y^2,\quad x^2+y^2=4 \Rightarrow \left.\begin{matrix} 2x=2x\lambda \\ -2y=2y\lambda \end{matrix}\right\} \Rightarrow$

$\lambda=0 \rightarrow x=y=0$ לא מקיים את האילוץ

$\lambda\neq 0 \rightarrow \begin{cases} \lambda=1 \rightarrow y=0,\ x=\pm 2 \\ \lambda\neq 1 \rightarrow x=0,\ y=\pm 2 \end{cases}$

נקודה	$(2,0)$	$(-2,0)$	$(0,2)$	$(0,-2)$
$f(x,y)$	4	4	-4	-4

$\Rightarrow f_{max}=4$

$f_{min}=-4$

② $f(x,y,z)=x^2+y^2+z^2,\quad 3x+2y+z=6$

$\Rightarrow \begin{cases} 2x=3\lambda \\ 2y=2\lambda \\ 2z=\lambda \end{cases} \rightarrow$

$\lambda=0 \rightarrow x=y=z=0$ לא האילוץ

$\lambda\neq 0 \rightarrow \left.\begin{matrix} \frac{2}{3}x=y=2z \\ 3x+2y+z=6 \end{matrix}\right\} \Rightarrow \left(\frac{9}{7},\frac{6}{7},\frac{3}{7}\right)$

הנקודה היא מינימום מכיוון שאם ניקח ערכים שליליים של x,y ו-z חיובי נקבל ערכי פונ' גדולים

③ $f(x,y,z)=x+y+z \quad , x^2+4y^2+9z^2=36$

$$\left.\begin{aligned} \lambda&=2x \\ \lambda&=8y \\ \lambda&=18z \end{aligned}\right\} \to \begin{aligned} &\lambda=0 \to x=y=z=0 \quad \text{לא מקיים} \\ &\lambda\neq 0 \to x=4y=9z \to \begin{aligned} x&=9z \\ y&=\tfrac{9}{4}z \end{aligned} \to \end{aligned}$$

$$(9z)^2+4\left(\frac{9}{4}z\right)^2+9z^2=36 \to z=\pm\frac{4}{7}$$

$$\to \begin{aligned} &\left(\frac{36}{7},\frac{9}{7},\frac{4}{7}\right) \\ &\left(-\frac{36}{7},-\frac{9}{7},-\frac{4}{7}\right) \end{aligned} \to \begin{aligned} &f\left(\frac{36}{7},\frac{9}{7},\frac{4}{7}\right)=7 \to \max \\ &f\left(-\frac{36}{7},-\frac{9}{7},-\frac{4}{7}\right)=-7 \to \min \end{aligned}$$

④ $f(x,y,z)=x^2+y^2+z^2 \quad , x+y+z=1 \quad , x+2y+3z=6$

$$\left.\begin{aligned} 2x&=\lambda+\mu \\ 2y&=\lambda+2\mu \\ 2z&=\lambda+3\mu \end{aligned}\right\} \Rightarrow \left.\begin{aligned} x&=\frac{\lambda+\mu}{2} \\ y&=\frac{\lambda+2\mu}{2} \\ z&=\frac{\lambda+3\mu}{2} \end{aligned}\right\} \to \begin{aligned} &\frac{\lambda+\mu}{2}+\frac{\lambda+2\mu}{2}+\frac{\lambda+3\mu}{2}=1 \\ &\frac{\lambda+\mu}{2}+2\left(\frac{\lambda+2\mu}{2}\right)+3\left(\frac{\lambda+3\mu}{2}\right)=6 \end{aligned} \to$$

$$\left.\begin{aligned} 3\lambda+6\mu&=2 \\ 6\lambda+14\mu&=12 \end{aligned}\right\} \to \left.\begin{aligned} \lambda&=-\frac{22}{3} \\ \mu&=4 \end{aligned}\right\} \to \begin{aligned} x&=-\frac{5}{3} \\ y&=\frac{1}{3} \\ z&=\frac{7}{3} \end{aligned}$$

$$\to \left(-\frac{5}{3},\frac{1}{3},\frac{7}{3}\right) \qquad f\left(-\frac{5}{3},\frac{1}{3},\frac{7}{3}\right)=\frac{25}{3} \to \min$$

⑥

$$\int_0^1\int_0^1 xy\,e^{x+y}\,dy\,dx = \int_0^1\int_0^1 xe^x ye^y\,dy\,dx =$$

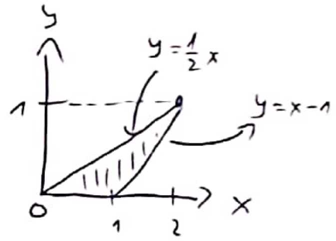
$$\int_0^1 xe^x \int_0^1 ye^y\,dy\,dx = \left[\ \int_0^1 ye^y\,dy = \begin{Bmatrix} u=y & u'=1 \\ v'=e^y & v=e^y \end{Bmatrix} =\right.$$

$$\left. ye^y\Big]_0^1 - \int_0^1 e^y\,dy = e - e^y\Big]_0^1 = e-(e-1) = 1\ \right] =$$

$$= \int_0^1 xe^x\cdot 1\,dx = \int_0^1 xe^x\,dx = 1$$

⑦

$$\int_0^1\int_{2y}^{y+1} (x-y)\,dx\,dy = \int \left[\frac{x^2}{2} - xy\right]_{2y}^{y+1} dy =$$



$y=\frac{1}{2} \Rightarrow x=2y$

$y=x-1 \Rightarrow x=y+1$

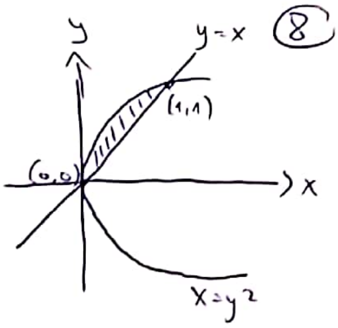
$$\int_0^1 \left[\frac{(y+1)^2}{2} - (y+1)\cdot y - \left(\frac{(2y)^2}{2} - 2y\cdot y\right)\right] dy =$$

$$= \int_0^1 \left[\frac{(y+1)^2}{2} - y^2 - y - \cancel{2y^2} + \cancel{2y^2}\right] dy = \int_0^1 \left(\frac{y^2}{2} + \cancel{y} + \frac{1}{2} - y^2 - \cancel{y}\right) dy$$

$$= \int_0^1 \left(\frac{1}{2} - \frac{y^2}{2}\right) dy = \left[\frac{1}{2}y - \frac{y^3}{6}\right]_0^1 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

המשך

8



$$\int_0^1\int_{y^2}^{y} dx\,dy = \int_0^1 [x]_{y^2}^{y}\, dy =$$

$$= \int_0^1 (y-y^2)\,dy = \left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1 = \frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

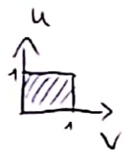
9

$$\iint_D (x^2-y^2)\,dx\,dy = \left(\begin{matrix} x=u+v\\ y=u-v\end{matrix}\ ,\ \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix}1&1\\1&-1\end{vmatrix} = -2\right)$$

$$=\iint_{D^*}\left[(u+v)^2-(u-v)^2\right]\cdot|-2|\,du\,dv =$$

$$(u,v)=\left(\frac{x+y}{2},\ \frac{x-y}{2}\right)$$

(x,y)	$\to$	(u,v)
$(0,0)$	$\to$	$(0,0)$
$(1,-1)$	$\to$	$(0,1)$
$(1,1)$	$\to$	$(1,0)$
$(2,0)$	$\to$	$(1,1)$



$$\iint_{D^*} 4uv\cdot 2\,du\,dv = 8\int_0^1\int_0^1 uv\,du\,dv = 8\cdot \frac{u^2}{2}\Big]_0^1\cdot\frac{v^2}{2}\Big]_0^1 =$$

$$8\cdot\frac{1}{2}\cdot\frac{1}{2}=\boxed{2}$$

10

$$\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)^{\frac{5}{2}}\,dy\,dx=\int_0^{\frac{\pi}{2}}\int_0^1(r^2)^{\frac{5}{2}}\, r\,dr\,d\theta=$$

$$=\int_0^{\frac{\pi}{2}}d\theta\int_0^1 r^6\,dr=\frac{\pi}{2}\cdot\left[\frac{r^7}{7}\right]_0^1=\frac{\pi}{14}$$