CHAPTER FIVE

The Logic of Relations

I. SYMBOLIZING RELATIONS

Some propositions which contain two or more proper names (of individuals) are correctly interpreted as truth-functional compounds of singular propositions having different subject terms. For example, the proposition

Lincoln and Grant were presidents.

is properly interpreted as the conjunction of the two singular propositions

Lincoln was a president and Grant was a president.

But for some other propositions having the same verbal pattern that analysis is wholly unsatisfactory. Thus the proposition

Lincoln and Grant were acquainted.

is definitely *not* a conjunction or any other truth function of the two expressions

Lincoln was acquainted and Grant was acquainted.

On the contrary, dividing the proposition in this way destroys its significance, for its meaning is not that both Lincoln and Grant were (or had) acquaintances, but that they were *acquainted with each other*. The given proposition does not assert that Lincoln and Grant both had a certain *property*, but that they stood in a certain *relationship*. Lincoln is not said simply to be acquainted (whatever that might mean), but *acquainted with Grant*. Other propositions which express relations between two individuals are

John loves Mary.
Plato was a student of Socrates.
Isaac was a son of Abraham.
New York is east of Chicago.
Chicago is smaller than New York.

Relations such as these, which can hold between two individuals, are called 'binary' or 'dyadic'. Other relations may relate three or more individuals. For example, the propositions

Detroit is between New York and Chicago.
Helen introduced John to Mary.
America won the Phillipines from Spain.

express *ternary* or *triadic* relations, while *quaternary* or *tetradic* relations are expressed by the propositions

America bought Alaska from Russia for seven million dollars.
Jack traded his cow to the peddler for a handful of beans.
Al, Bill, Charlie, and Doug played bridge together.

Relations enter into arguments in various ways. One example of a relational argument is

Al is older than Bill.
Bill is older than Charlie.
Therefore, Al is older than Charlie.

A slightly more complex example, which involves quantification, is this:

Helen likes David.
Whoever likes David likes Tom.
Helen likes only good-looking men.
Therefore, Tom is a good-looking man.

A still more complex relational inference, which involves multiple quantification, is:

All horses are animals.
Therefore, the head of a horse is the head of an animal.

The latter is a valid inference which, as De Morgan observed, all the logic of Aristotle will not permit one to draw. Its validation by our apparatus of quantifiers and propositional functions will be set forth in the next section.

Before discussing the validation of relational arguments, which will require no methods of proof beyond those developed in the preceding chapter, the problem of *symbolizing* relational propositions must be dealt with. Just as a single predicate symbol can occur in different propositions, so a single relation symbol can occur in different propositions. Just as we have the predicate 'human' common to the propositions:

Aristotle is human.
Plato is human.
Socrates is human.

so we have the relational word 'teacher' common to the propositions:

Socrates was a teacher of Plato.
Plato was a teacher of Aristotle.

And just as we regard the three subject-predicate propositions as different substitution instances of the propositional function '*x* is human', so we can regard the two relational propositions as different substitution instances of the propositional function '*x* was a teacher of *y*'. Replacing the variable '*x*' by the constant 'Socrates' and the variable '*y*' by the constant 'Plato' gives us the first proposition; replacing the '*x*' by 'Plato' and the '*y*' by 'Aristotle' gives the second. The *order* of replacement is of great importance here: if '*x*' is replaced by 'Aristotle' and '*y*' by 'Plato', the result is the *false* proposition

Aristotle was a teacher of Plato.

Just as a propositional function of one variable like '*x* is human' was abbreviated as '*Hx*', so a propositional function of two variables like '*x* was the teacher of *y*' is abbreviated as '*Txy*'. Similarly, the propositional function '*x* is between *y* and *z*' will be abbreviated as '*Bxyz*', and the propositional function '*x* traded *y* to *z* for *w*' will be abbreviated as '*Txyzw*'. Our first specimen of a relational argument, since it involves no quantifications, is very easily symbolized. Using the individual constants '*a*', '*b*', and '*c*' to denote Al, Bill, and Charlie, and the expression '*Oxy*' to abbreviate '*x* is older than *y*', we have

Oab
Obc
$\therefore Oac$

Our second argument is not much more difficult, since none of its propositions contains more than a single quantification. Using the individual constants '*h*', '*d*', and '*t*' to denote Helen, David, and Tom, respectively, '*Gx*' to abbreviate '*x* is a good-looking man', and the symbol '*Lxy*' to abbreviate '*x* likes *y*', the argument can be symbolized as

1. Lhd
2. $(x)(Lxd \supset Lxt)$
3. $(x)(Lhx \supset Gx)$

$\therefore Gt$

The demonstration of its validity is so easily constructed that it may well be set down now, before going on to consider some of the more difficult problems of symbolization. Referring back to the numbered premisses above, the demonstration proceeds:

4. $Lhd \supset Lht$	2, **UI**
5. Lht	4, 1, M.P.
6. $Lht \supset Gt$	3, **UI**
7. Gt	6, 5, M.P.

Symbolizing relational propositions becomes more complicated when several quantifications occur in a single proposition. Our discussion of the problem will be simplified by confining attention at first to two individual constants, '*a*', and '*b*', and the propositional function '*x* attracts *y*', abbreviated as '*Axy*'. The two statements '*a* attracts *b*' and '*b* is attracted by *a*' obviously have the same meaning, the first expressing that meaning by use of the *active voice*, the second by use of the *passive voice*. Both statements translate directly into the single formula '*Aab*'. Similarly, the two statements '*b* attracts *a*' and '*a* is attracted by *b*' are both symbolized by the formula '*Aba*'. These two different substitution instances of '*Axy*' are logically independent of each other, either can be true without entailing the truth of the other.

We are still on elementary and familiar ground when we come to symbolize

'*a* attracts everything'
'everything is attracted by *a*' } as '$(x)Aax$',
'*a* attracts something'
'something is attracted by *a*' } as '$(\exists x)Aax$',
'everything attracts *a*'
'*a* is attracted by everything' } as '$(x)Axa$',
'something attracts *a*'
'*a* is attracted by something' } as '$(\exists x)Axa$'.

But the problem of symbolizing becomes more complex when we dispense entirely with individual constants and consider relational propositions which are completely general. The simplest propositions of this kind are

1. Everything attracts everything.
2. Everything is attracted by everything.
3. Something attracts something.
4. Something is attracted by something.
5. Nothing attracts anything.
6. Nothing is attracted by anything.

which are symbolized by the following formulas:

1. $(x)(y)Axy$
2. $(y)(x)Axy$
3. $(\exists x)(\exists y)Axy$
4. $(\exists y)(\exists x)Axy$
5. $(x)(y) \sim Axy$
6. $(y)(x) \sim Axy$

In their English formulations, propositions 1 and 2 are clearly equivalent to each other, as are 3 and 4, and 5 and 6. The first two equivalences are easily established for the corresponding logical formulas:

→1. $(x)(y)Axy$	
2. $(y)Awy$	1, **UI**
3. Awv	2, **UI**
4. $(x)Axv$	3, **UG**
5. $(y)(x)Axy$	4, **UG**
6. $(x)(y)Axy \supset (y)(x)Axy$	1–5, C.P.

→1. $(\exists x)(\exists y)Axy$	
2. $(\exists y)Awy$	1, **EI**
3. Awv	2, **EI**
4. $(\exists x)Axv$	3, **EG**
5. $(\exists y)(\exists x)Axy$	4, **EG**
6. $(\exists x)(\exists y)Axy \supset (\exists y)(\exists x)Axy$	1–5, C.P.

These demonstrate the logical truth of conditionals rather than of equivalences, but that their converses are true also can be established by simply reversing the orders of steps 1 through 5. (The equivalence between formulas 5 and 6 is clearly established by the same pattern of argument that proves 1 equivalent to 2.)

When we turn to the next pair of statements

7. Everything attracts something.
8. Something is attracted by everything.

there is no longer any logical equivalence or sameness of meaning. Sentence 7 is not entirely unambiguous, and some exceptional contexts might shift its meaning, but its most natural interpretation is *not* that there is some one thing which is attracted by everything, but rather that everything attracts *something or other*. We can approach its symbolization by way of successive paraphrasings, writing first

$(x)(x \text{ attracts something})$

and then symbolizing the expression '*x* attracts something' the same way in which we symbolized '*a* attracts something'. This gives us the formula

7. $(x)(\exists y)Axy$.

Sentence 8 is also susceptible of alternative interpretations, one of which would make it synonymous with sentence 7, meaning that something or other is attracted by any (given) thing. But a perfectly straightforward way of understanding sentence 8 is to take it as asserting that some *one thing* is attracted by all things. Its symbolization, too, can be accomplished in a stepwise fashion, writing first

$(\exists y)(y \text{ is attracted by everything})$

and then symbolizing the expression '*y* is attracted by everything' the same way in which we symbolized '*a* is attracted by everything'. This gives us the formula

8. $(\exists y)(x)Axy$.

There is a certain *misleading* similarity between formulas 7 and 8. They both consist of the propositional function '*Axy*' to which are applied a universal quantifier with respect to '*x*' and an existential quantifier with respect to '*y*'. But the *order* in which the quantifiers are written is different in each case, and that makes a world of difference in their meanings. Formula 7, in which the universal quantifier comes first, asserts that given anything in the universe, there is something or other which it attracts. But formula 8, in which the existential quantifier comes

first, asserts that there is some one thing in the universe such that everything in the universe attracts *it*. Where two quantifiers are applied to one propositional function, if they are both universal or both existential, their order does not matter, as is shown by the equivalence of formulas 1 and 2, 3 and 4, and 5 and 6. But where one is universal and the other existential the order of generalization or quantification is very important indeed.

Although formulas 7 and 8 are not equivalent, they are not independent. The former is validly deducible from the latter. The demonstration is easily constructed as follows:

1.	$(\exists y)(x)Axy$	
2.	$(x)Axv$	1, **EI**
3.	Auv	2, **UI**
4.	$(\exists y)Auy$	3, **EG**
5.	$(x)(\exists y)Axy$	4, **UG**

But the inference is valid only one way. Any attempt to derive formula 8 from 7 must inevitably run afoul of one of the restrictions on **UG**.

A similar pair of inequivalent propositions may be written as

9. Everything is attracted by something.
10. Something attracts everything.

These are clearly inequivalent when the 'something' in 9, coming at the end, is understood as 'something or other', and the 'something' in 10, coming at the beginning, is understood as 'some one thing'. They are symbolized as

9. $(y)(\exists x)Axy$.
10. $(\exists x)(y)Axy$.

Relational propositions are sometimes formulated as though they were simple subject-predicate assertions. For example, '*a* was struck' is most plausibly interpreted to assert that *something struck a*. Such implicit occurrences of relations are often marked by the passive voice of a transitive verb. Our symbolization of propositions containing implicit relations should be guided by consideration of the use to which they are to be put. Our motive in symbolizing arguments is to get them into that form which is most convenient for testing their validity by the application of our rules. Our goal, therefore, with respect to a given argument, is not that of providing a theoretically complete analysis, but rather of providing one sufficiently complete for the purpose at hand—the testing of validity. Consequently some implicit relations may be left implicit, while others require a more thorough analysis, as may be made clear by an example. Consider the argument

> Whoever visited the building was observed. Anyone who had observed Andrews would have remembered him. Nobody remembered Andrews. Therefore, Andrews didn't visit the building.

The first proposition of this argument contains two relations, one explicit, the other implicit. Explicitly, we have the relation of *someone visiting the building*. It is explicit because mention is made both of the visitor and what was visited by him. Implicitly, we have the relation of *someone observing someone*, which is implicit because no mention is made of the someone who does the observing—the omission being marked by the use of the passive voice. However, because the only other occurrence of '*x* visited the building' is also as a *unit*, in the conclusion, it need not be treated as a relation at all, but may be symbolized as a simple predicate. On the other hand, '*x* observed *y*', despite its merely implicit occurrence in the first premiss, must be explicitly symbolized as a relation if the validity of the argument is to be proved. For its second occurrence is not a simple repetition of the original unit; it appears instead as an explicit relation, with the first variable quantified and the second replaced by the proper name 'Andrews'. Using '*a*' to denote Andrews, '*Vx*' to abbreviate '*x* visited the building', '*Oxy*' to abbreviate '*x* observed *y*', and '*Rxy*' to abbreviate '*x* remembers *y*', a symbolic translation and validation of the given argument may be written as

1.	$(x)[Vx \supset (\exists y)Oyx]$	
2.	$(x)[Oxa \supset Rxa]$	
3.	$(x) \sim Rxa$	$/ \therefore \sim Va$
4.	$Oza \supset Rza$	2, UI
5.	$\sim Rza$	3, UI
6.	$\sim Oza$	4, 5, M.T.
7.	$(y) \sim Oya$	6, UG
8.	$\sim(\exists y)Oya$	7, QN
9.	$Va \supset (\exists y)Oya$	1, UI
10.	$\sim Va$	9, 8, M.T.

Our demonstration of the validity of this argument would not have been helped at all by symbolizing 'Andrews visited the building' as a substitution instance of the relational '*x* visited *y*' rather than of the simpler '*Vx*'. But our demonstration absolutely required us to symbolize 'was observed' explicitly as a relation.

While on the subject of implicit or concealed relations, mention must be made of the philosophically interesting but logically troublesome topic of *pseudo-relations*. Examples of these are *desiring*, *hoping*, *planning*, *wishing-for*, and the like. These can be regarded as *pseudo-relations* because of the fact that certain inferences which are valid in connection with ordinary relations break down or are invalid when made with respect to *apparent* relations of the sort mentioned. If I *attend* a picnic, there must exist a picnic for me to attend. But if I merely *plan* a picnic, and never execute my plans, there need not exist any picnic at all. If I marry a perfect wife, there must exist a perfect wife for me to marry. But if I merely *desire* a perfect wife, it by no means follows that there exists a perfect wife to whom I stand in the relation of desiring. The existence of Santa Claus is not established by believing in him, for *believing in* is a pseudo rather than a genuine relation. We must beware of imputing existence to non-existents by mistaking pseudo-relations for genuine ones.

Most of our previous examples were illustrations of *unlimited* generality, in which it was asserted that *everything* stood in such-and-such a relation, or something did, or nothing did. A great many relational propositions are not so sweeping. Most assertions are more modest, claiming not that *everything* stands in such-and-such a relation, but that everything does *if* it satisfies certain conditions or restrictions. Thus we may say either that

Everything is attracted by all magnets.

or that

Everything made of iron is attracted by all magnets.

The second, of course, is the more modest assertion, being less *general* than the first. While the first is adequately symbolized, where '*Mx*' abbreviates '*x* is a magnet', as

$$(x)(y)[My \supset Ayx],$$

the second is symbolized, where '*Ix*' abbreviates '*x* is made of iron', as

$$(x)[Ix \supset (y)(My \supset Ayx)].$$

That the symbolization is correct can be seen by paraphrasing the second proposition in English as

Given anything at all, *if* it is made of iron then it is attracted by all magnets.

Perhaps the best way to symbolize relational propositions is by the kind of stepwise process that has already been exemplified. Let us illustrate it further, this time for propositions of limited generality. First let us consider the proposition

Any good amateur can beat some professional.

As a first step we may write

$(x)\{(x$ is a good amateur$) \supset (x$ can beat some professional$)\}$.

Next, the consequent of the conditional between the braces

x can beat some professional

is symbolized as a generalization or quantified expression:

$(\exists y)[(y$ is a professional$)\cdot(x$ can beat $y)]$.

Now, using the obvious abbreviations, '*Gx*', '*Px*', and '*Bxy*' for '*x* is a good amateur', '*x* is a professional', and '*x* can beat *y*', the given proposition is symbolized by the formula

$$(x)[Gx \supset (\exists y)(Py\cdot Bxy)].$$

Using the same method of paraphrasing by successive steps, we may symbolize

Some professionals can beat all amateurs.

first as

$(\exists x)[(x$ is a professional$)\cdot(x$ can beat all amateurs$)]$

then as

$(\exists x)\{(x$ is a professional$)\cdot(y)[(y$ is an amateur$) \supset (x$ can beat $y)]\}$

and finally (using abbreviations) as

$$(\exists x)[Px\cdot(y)(Ay \supset Bxy)].$$

The same method is applicable in more complex cases, where more than one relation is involved. We symbolize the proposition

Anyone who promises everything to everyone is certain to disappoint somebody.

first by paraphrasing it as

$(x)\{[(x$ is a person$)\cdot(x$ promises everything to everyone$)]$
$\supset [x$ disappoints somebody$]\}$.

The second conjunct of the antecedent

x promises everything to everyone

may be further paraphrased, first as

$(y)[(y$ is a person$) \supset (x$ promises everything to $y)]$

and then as

$(y)[(y$ is a person$) \supset (z)(x$ promises z to $y)]$.

The consequent in our first paraphrase

x disappoints somebody

has its structure made more explicit by being rewritten as

$$(\exists u)[(u \text{ is a person})\cdot(x \text{ disappoints } u)].$$

The original proposition can now be rewritten as

$$(x)\{\{(x \text{ is a person})\cdot(y)[(y \text{ is a person}) \supset (z)(x \text{ promises } z \text{ to } y)]\} \supset (\exists u)[(u \text{ is a person})\cdot(x \text{ disappoints } u)]\}.$$

Using the obvious abbreviations, '*Px*', '*Pxyz*', '*Dxy*' for '*x* is a person', '*x* promises *y* to *z*', and '*x* disappoints *y*', the proposition can be expressed more compactly in the formula

$$(x)\{\{Px\cdot(y)[Py \supset (z)Pxzy]\} \supset (\exists u)(Pu\cdot Dxu)\}.$$

With practice, of course, not all such intermediate steps need be written cut explicitly.

Quantification words such as 'everyone', 'anyone', 'everybody', 'anybody', and 'whoever', refer to *all persons* rather than to *all things;* and such quantification words as 'someone' and 'somebody' refer to *some persons* rather than to *some things.* It is frequently desirable to represent this reference in our symbolization. But doing so is not always necessary for the purpose of evaluating arguments containing these words, however, and the choice of symbolization procedure is determined on the same grounds on which one decides whether a relational clause or phrase is to be symbolized explicitly as a relation or as a mere predicate.

The words 'always', 'never', and 'sometimes' frequently have a strictly non-temporal significance, as in the propositions

> Good men always have friends.
> Bad men never have friends.
> Men who have no wives sometimes have friends.

which may be symbolized, using obvious abbreviations, as

$$(x)[(Gx\cdot Mx) \supset (\exists y)Fxy]$$
$$(x)[(Bx\cdot Mx) \supset \sim(\exists y)Fxy]$$
$$(\exists x)\{[Mx\cdot\sim(\exists y)(Wy\cdot Hxy)]\cdot(\exists z)Fxz\}.$$

However, some uses of these words are definitely temporal, and when they are, they can be symbolized by the logical machinery already available, as can other temporal words like 'while', 'when', 'whenever', and the like. An example or two should serve to make this clear. Thus the proposition

> Dick always writes Joan when they are separated.

asserts that all times when Dick and Joan are separated are times when Dick writes Joan. This can be symbolized using '*Tx*' for '*x* is a time', '*Wxyz*' for '*x* writes *y* at (time) *z*', and '*Sxyz*' for '*x* and *y* are separated at (time) *z*', as

$$(x)\{Tx \supset [Sdjx \supset Wdjx]\}$$

Perhaps the most vivid illustration of the adaptability of the present notation is in symbolizing the following remark, usually attributed to Lincoln:

> You can fool some of the people all of the time, and all of the people some of the time, but you cannot fool all of the people all of the time.

The first conjunct: 'You can fool some of the people all of the time' is ambiguous. It may be taken to mean either that *there is at least one person who can always be fooled* or that *for any time there is at least one person (or other) who can be fooled at that time.* Adopting the first interpretation, and using '*Px*' for '*x* is a person', '*Tx*' for '*x* is a time', and '*Fxy*' for 'you can fool *x* at (or during) *y*', the above may be symbolized as

$$\{(\exists x)[Px\cdot(y)(Ty \supset Fxy)]\cdot(\exists y)[Ty\cdot(x)(Px \supset Fxy)]\}\cdot(\exists y)(\exists x)[Ty\cdot Px\cdot\sim Fxy)].$$

The actual testing of relational arguments presents no new problems—once the translations into logical symbolism are effected. The latter is the more troublesome part, and so a number of exercises are provided for the student to do before going on.