1 The set $X \subset \mathbb{A}^2$ is defined by the equation $f : x^2 + y^2 = 1$ and $g : x = 1$. Find the ideal $\mathfrak{A}_X$. Is it true that $\mathfrak{A}_X = (f, g)$?

2 Let $X \subset \mathbb{A}^2$ be the algebraic plane curve defined by $y^2 = x^3$. Prove that an element of $k[X]$ can be written uniquely in the form $P(x) + Q(x)y$ with $P(x)$, $Q(x)$ polynomials.

3 Let X be the curve of Exercise 2 and $f(t) = (t^2, t^3)$ the regular map $\mathbb{A}^1 \to X$. Prove that f is not an isomorphism. [Hint: Try to construct the inverse of f as a regular map, using the result of Exercise 2.]

4 Let X be the curve defined by the equation $y^2 = x^2 + x^3$ and $f: \mathbb{A}^1 \to X$ the map defined by $f(t) = (t^2 - 1, t(t^2 - 1))$. Prove that the corresponding homomorphism f^* maps $k[X]$ isomorphically to the subring of the polynomial ring $k[t]$ consisting of polynomials $g(t)$ such that $g(1) = g(-1)$. (Assume that $\operatorname{char} k \neq 2$.)

5 Prove that the hyperbola defined by $xy = 1$ and the line $\mathbb{A}^1$ are not isomorphic.

6 Consider the regular map $f: \mathbb{A}^2 \to \mathbb{A}^2$ defined by $f(x, y) = (x, xy)$. Find the image $f(\mathbb{A}^2)$; is it open in $\mathbb{A}^2$? Is it dense? Is it closed?

7 The same question as in Exercise 6 for the map $f: \mathbb{A}^3 \to \mathbb{A}^3$ defined by $f(x, y, z) = (x, xy, xyz)$.

8 An isomorphism $f: X \to X$ of a closed set X to itself is called an *automorphism*. Prove that all automorphisms of the line $\mathbb{A}^1$ are of the form $f(x) = ax + b$ with $a \neq 0$.

9 Prove that the map $f(x, y) = (\alpha x, \beta y + P(x))$ is an automorphism of $\mathbb{A}^2$, where $\alpha, \beta \in k$ are nonzero elements, and $P(x)$ is a polynomial. Prove that maps of this type form a group B.

10 Prove that if $f(x_1, \dots, x_n) = (P_1(x_1, \dots, x_n), \dots, P_n(x_1, \dots, x_n))$ is an automorphism of $\mathbb{A}^n$ then the Jacobian $J(f) = \det|\frac{\partial P_i}{\partial x_j}| \in k$. Prove that $f \mapsto J(f)$ is a homomorphism from the group of automorphisms of $\mathbb{A}^n$ into the multiplicative group of nonzero elements of k.

11 Suppose that X consists of two points. Prove that the coordinate ring $k[X]$ is isomorphic to the direct sum of two copies of k.

12 Let $f: X \to Y$ be a regular map. The subset $\Gamma_f \subset X \times Y$ consisting of all points of the form $(x, f(x))$ is called the *graph* of f. Prove that (a) $\Gamma_f \subset X \times Y$ is a closed subset, and (b) Γ_f is isomorphic to X.

13 The map $p_Y: X \times Y \to Y$ defined by $p_Y(x, y) = y$ is called the *projection* to Y or the *second projection*. Prove that if $Z \subset X$ and $f: X \to Y$ is a regular map then

$f(Z) = p_Y((Z \times Y) \cap \Gamma_f)$, where Γ_f is the graph of f and $Z \times Y \subset X \times Y$ is the subset of (z, y) with $z \in Z$.

14 Prove that for any regular map $f: X \to Y$ there exists a regular map $g: X \to X \times Y$ that is an isomorphism of X with a closed subset of $X \times Y$ and such that $f = p_Y \circ g$. In other words, any map is the composite of an embedding and a projection.

15 Prove that if $X = \bigcup U_\alpha$ is any covering of a closed set X by open subsets U_α then there exists a finite number $U_{\alpha_1}, \dots, U_{\alpha_r}$ of the U_α such that $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_r}$.

16 Prove that the Frobenius map φ (Example 1.16) is a one-to-one correspondence. Is it an isomorphism, for example if $X = \mathbb{A}^1$?

17 Find the zeta function $Z_X(t)$ for $X = \mathbb{A}^n$.

18 Determine $Z_X(t)$ for X a nonsingular conic in $\mathbb{A}^2$.