תרגיל 2

להגשה עד 7.4.2014, ז' ניסן. כל אחד בקבוצת התירגול שלו

פשוטים

1. $\int \sqrt{x} dx$
2. $\int \frac{x+1}{\sqrt{x}} dx$
3. $\int (x^2+1)^2 dx$

הצבה

1. $\int \frac{e^x}{e^{2x}+1} dx$
2. $\int \frac{1}{x\ln^3(x)} dx$
3. $\int \frac{2x^3}{\sqrt{x^2+1}} dx$

בחלקים:

1. $\int x^n \ln(x),\ \ n \neq -1$
2. $\int x \cdot \ln((x-2)^{1/3})$
3. $\int \frac{x}{\cos^2(x)} dx$

טריגונומטרי

1. $\int \frac{\cos x}{\sqrt[3]{\sin^2(x)}} dx$
2. $\int \frac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)} dx$

פונקציות רציונאליות

1. $\int \frac{x^5}{x^2+2} dx$
2. $\int \frac{x^2+6}{x(x-3)^2} dx$

כללי

1. $\int e^{2\sin(x)}\cos(x)dx$
2. $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx$
3. $\int \frac{1}{\sqrt{x^2-1}}dx$
4. $\int \sqrt{1+\frac{1}{x^2}}dx$
5. $\int e^{2x}\sin^2(x)dx$
6. $\int \frac{1}{x^3+2x^2+x+2}dx$
7. $\int x\tan^2(x)dx$
8. $\int \frac{-2x+7}{\sqrt{(-x^2+7x-10)^3}}dx$
9. $\int \frac{1}{\sqrt{x^2+1}}dx$

פתרון

פשוטים

1. $\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + c$
2. $\int \frac{x+1}{\sqrt{x}}dx = \int \sqrt{x} + \frac{1}{\sqrt{x}}dx = \frac{2}{3}x^{\frac{3}{2}} + 2\sqrt{x} + c$
3. $\int (x^2+1)^2dx = \int (x^4+2x^2+1)dx = \frac{x^5}{5} + 2\frac{x^3}{3} + x + c$

הצבה

1. $\int \frac{e^x}{e^{2x}+1}dx = [t=e^x] = \int \frac{1}{t^2+1}dt = \arctan(t)+c = \arctan(e^x)+c$
2. $\int \frac{1}{x\ln^3(x)}dx = [t=\ln(x) \Rightarrow dt = \frac{1}{x}dx] = \int \frac{1}{t^3}dt = -\frac{1}{t^2}+c = -\frac{1}{\ln^2(x)}+c$
3. $\int \frac{2x^3}{\sqrt{x^2+1}}dx = [t=x^2+1 \Rightarrow dt = 2xdx] = \int \frac{t-1}{\sqrt{t}}dt =$
   $= [as\, before] = \frac{2}{3}t^{\frac{3}{2}} - 2\sqrt{t} + c = \frac{2}{3}(x^2+1)^{\frac{3}{2}} - 2\sqrt{(x^2+1)} + c$

בחלקים:

1. $\int x^n\ln(x),\ n\neq -1 = [f'(x)=x^n,\, g(x)=\ln(x)] = \frac{x^{n+1}}{n+1}\ln(x) - \int \frac{x^{n+1}}{n+1}\cdot \frac{1}{x}dx$
   $= \frac{x^{n+1}}{n+1}\ln(x) - \frac{x^{n+1}}{(n+1)^2} + c$

.2 $\int x\cdot\ln((x-2)^{1/3}) = [f'=x,\ g=\ln((x-2)^{1/3})]$
$= \frac{x^2}{2}\ln((x-2)^{1/3} - \int \frac{x^2}{2}\frac{\frac{1}{3}(x-2)^{-2/3}}{(x-2)^{1/3}}dx$
$= \frac{x^2}{2}\ln((x-2)^{1/3} - \frac{1}{6}\int\frac{x^2}{(x-2)}dx = \frac{x^2}{2}\ln((x-2)^{1/3} - \frac{1}{6}\int\frac{(x^2-2)+2}{(x-2)}dx$
$= \frac{x^2}{2}\ln((x-2)^{1/3} - \frac{1}{6}(x+2\ln|x-2|)+c$

.3 $\int\frac{x}{\cos^2(x)}dx = [f'=\frac{1}{\cos^2(x)}, g=x] = x\tan(x) - \int\tan(x)dx = [t=\cos(x)]$
$= x\tan(x) + \int\frac{1}{t}dt = x\tan(x) + \ln|\cos(x)| + c$

טריגונומטרי

.1 $\int\frac{\cos x}{\sqrt[3]{\sin^2(x)}}dx = [t=\sin(x)] = \int\frac{dt}{t^{2/3}} = 3t^{1/3}+c = 3(\sin(x))^{1/3}+c$

.2 $\int\frac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)}dx = \int\frac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)}\cdot\frac{\sin(x)-\cos(x)}{\sin(x)-\cos(x)}dx = \int\frac{(\sin(x)-\cos(x))^2}{\sin^2(x)-\cos^2(x)}dx$
$= \int\frac{1-\sin(2x)}{-\cos(2x)}dx = [t=\sin(2x)\Rightarrow dt = 2\cos(2x)dx] = \frac{1}{2}\int\frac{t-1}{\sqrt{1-t^2}}dt$
$= \frac{1}{2}(-\sqrt{1-t^2} - \arcsin(t)) + c = -\frac{1}{2}(\sqrt{1-\sin^2(x)} + \arcsin(\sin(x))) + c$

פונקציות רציונאליות

.1 $\int\frac{x^5}{x^2+2}dx = \int x^3 - 2x + \frac{4x}{x^2+2}dx = \frac{x^4}{4} - x^2 + 2\ln|x^2+2|$

.2 $\int\frac{x^2+6}{x(x-3)^2}dx = \int\frac{2}{3x} + `\frac{1}{3(x-3)} + \frac{5}{(x-3)^2}dx = \frac{2}{3}\ln|x| + \frac{1}{3}\ln|x-3| - 5\frac{1}{x-3}$

כללי

.1 $\int e^{2\sin(x)}\cos(x)dx = [t=\sin(x)] = \int e^{2t}dt = \frac{1}{2}e^{2t} = \frac{1}{2}e^{\sin(x)}$

.2 $\int\frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx = [x=t^6\Rightarrow dx = 6t^5dt] = \int\frac{t^6+t^4+t}{t^6(1+t^2)}dt =$
$\int\frac{1}{t^5} - \frac{1}{t^3} + \frac{1}{t^2} + \frac{1}{t} - \frac{t}{t^2+1}dt = -\frac{1}{t^4} + \frac{1}{t^2} - \frac{1}{t} + \ln|t| - \frac{1}{2}\ln|t^2+1|$
$-\frac{1}{(\sqrt[6]{x})^4} + \frac{1}{(\sqrt[6]{x})^2} - \frac{1}{(\sqrt[6]{x})} + \ln|\sqrt[6]{x}| - \frac{1}{2}\ln|(\sqrt[6]{x})^2+1|$

.3 $\int\frac{1}{\sqrt{x^2-1}}dx$ נציב $\sqrt{x^2-1} = t+x$
$\Rightarrow x^2-1 = t^2+2tx+x^2 \Rightarrow x = -\frac{t^2+1}{2t} = -\frac{1}{2}(t+\frac{1}{t})$
$\Rightarrow dx = -\frac{1}{2}(1-\frac{1}{t^2})dt = -\frac{1}{2}(\frac{(t-1)(t+1)}{t^2})dt$
$\int\frac{1}{\sqrt{x^2-1}}dx = \int\frac{1}{t+x}dx = \int\frac{1}{t-\frac{t^2+1}{2t}}(-\frac{1}{2}(\frac{(t-1)(t+1)}{t^2}))dt =$
$-\frac{1}{2}\int\frac{2t}{t^2-1}\frac{(t-1)(t+1)}{t^2}dt = -\int\frac{1}{t}dt = -\ln|t| = -\ln|\sqrt{x^2-1}-x|$

.4 $\int\sqrt{1+\frac{1}{x^2}}dx = \int\sqrt{\frac{x^2+1}{x^2}}dt$ נציב $\sqrt{x^2+1} = t+x$
$\Rightarrow x^2+1 = t^2+2tx+x^2 \Rightarrow x = \frac{1-t^2}{2t} = \frac{1}{2}(\frac{1}{t}-t)$
$\Rightarrow dx = \frac{1}{2}(-\frac{1}{t^2}-1)dt = -\frac{1}{2}(\frac{t^2+1}{t^2})dt,$
$\int\sqrt{\frac{x^2+1}{x^2}}dt = \int\frac{t+x}{x}dx = \int(\frac{t}{x}+1)dx = \int(\frac{2t^2}{1-t^2}+1)(-\frac{1}{2}(\frac{t^2+1}{t^2}))dt$
$= -\frac{1}{2}\int\frac{1+t^2}{1-t^2}\frac{t^2+1}{t^2}dt = -\frac{1}{2}\int\frac{t^4+2t^2+1}{t^2(1-t)(1+t)}dt$

$= -\frac{1}{2}\int -t + \frac{2}{t+1} - \frac{2}{t-1} + \frac{1}{t^2}dt = -\frac{1}{2}(-\frac{t^2}{2} + 2\ln|t+1| - 2\ln|t-1| - \frac{1}{t}) =$
$-\frac{1}{2}(-\frac{(\sqrt{x^2+1}-x)^2}{2} + 2\ln|\sqrt{x^2+1}-x+1| - 2\ln|\sqrt{x^2+1}-x-1| - \frac{1}{\sqrt{x^2+1}-x})$

.5 $\int e^{2x}\sin^2(x)dx = \int e^{2x}(\frac{\cos(2x)+1}{2})dx = \frac{1}{2}\int e^{2x} + e^{2x}\cos(2x)dx$
$= \frac{1}{2}(\frac{e^{2x}}{2} + \int e^{2x}\cos(2x)dx)$
נמשיך רק עם $\int e^{2x}\cos(2x)dx$
$\int e^{2x}\cos(2x)dx = [t = 2x] = \int e^t\cos(t)\frac{dt}{2}$
בתירגול ראינו כי
$F(x) = \int e^x\cos(x)dx = [g'(x) = \cos(x), f(x) = e^x] = e^x\sin(x) - \int e^x\sin(x)dx$
$= [g'(x) = \sin(x), f(x) = e^x] = e^x\sin(x) - \left(-e^x\cos(x) - \int -e^x\cos(x)dx\right) =$
$e^x(\sin(x) + \cos(x)) - F(x)$
ומכאן ש $2F(x) = e^x(\sin(x) + \cos(x))$ ולכן $F(x) = \frac{e^x(\sin(x)+\cos(x))}{2}$

ולכן נקבל ש
$\int e^{2x}\sin^2(x)dx = \frac{1}{2}(\frac{e^{2x}}{2} + \int e^{2x}\cos(2x)dx) = \frac{1}{2}(\frac{e^{2x}}{2} + \int e^t\cos(t)\frac{dt}{2})$
$= \frac{1}{4}(e^{2x} + \int e^t\cos(t)dt) = \frac{1}{4}(e^{2x} + \frac{e^t(\sin(t)+\cos(t))}{2}) =$
$= \frac{1}{4}(e^{2x} + \frac{e^{2x}(\sin(2x)+\cos(2x))}{2}) = \frac{1}{8}e^{2x}(\sin(2x) + \cos(2x) + 2)$

.6 $\int \frac{1}{x^3+2x^2+x+2}dx = \int \frac{1}{x^2(x+2)+(x+2)}dx = \int \frac{1}{(x^2+1)(x+2)}dx$
$= \int \frac{2-x}{5(x^2+1)} + \frac{1}{5(x+2)}dx = \int \frac{2}{5(x^2+1)} + \frac{2x}{-10(x^2+1)} + \frac{1}{5(x+2)}dx =$
$\frac{2}{5}\arctan(x) - \frac{1}{10}\ln|x^2+1| + \frac{1}{5}\ln|x+2|$

.7 $\int x\tan^2(x)dx = \int \frac{x\sin^2 x}{\cos^2 x}dx == [g' = \frac{1}{\cos^2(x)}, f = x\sin^2 x] =$
$x\sin^2(x)\tan(x) - \int \tan(x)(\sin^2 x + 2x\sin(x)\cos(x))dx$
נמשיך עם
$\int \tan(x)(\sin^2 x + 2x\sin(x)\cos(x))dx$
$= \int \frac{\sin^3 x}{\cos x} + 2x\sin^2(x)dx$
באשר ל $\int \frac{\sin^3 x}{\cos x}dx$ נפתור בהצבה
$\int \frac{\sin^3 x}{\cos x}dx = [t = \cos(x)] = \int \frac{1-t^2}{t}(-dt) =$
$-\int \frac{1}{t} - tdt = -\ln|t| + \frac{t^2}{2} = -\ln|\cos(x)| + \frac{\cos^2(x)}{2}$
באשר ל $\int 2x\sin^2(x)dx$
$= \int x(\cos(2x)+1)dx = \frac{x^2}{2} + \int x\cos(2x)dx = [f = x, g' = \cos(2x)]$
$= \frac{x^2}{2} + x\frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2}dx = \frac{x^2}{2} + x\frac{\sin(2x)}{2} + \frac{\cos(2x)}{4}$
תשובה סופית
$\int x\tan^2(x)dx = x\sin^2(x)\tan(x) - [-\ln|\cos(x)| + \frac{\cos^2(x)}{2} + \frac{x^2}{2} + x\frac{\sin(2x)}{2} + \frac{\cos(2x)}{4}]$

.8 $\int \frac{-2x+7}{\sqrt{(-x^2+7x-10)^3}}dx = \int(-x^2+7x-10)'(-x^2+7x-10)^{-3/2}dx = -2(-x^2+7x-10)^{-1/2}$

.9 $\int \frac{1}{\sqrt{x^2+1}}dx = [x = \sinh(t) \Rightarrow dx = \cosh(t)dt]$

$$= \int \frac{1}{\sqrt{1+\sinh^2(t)}} \cosh(t) dt = \int \frac{1}{\cosh(t)} \cosh(t) dt = \int 1 dt$$
$$= t = \sinh^{-1}(x)$$