**Notes of lecture 1.**

## 1. Affine algebraic varieties.

**Definition**. 1. An affine algebraic variety $\mathcal{X}$ (over a field $k$) is a pair $\mathcal{X} = (X, A)$, where $X$ is a set and $A \subset F(X)$ is a $k$-subalgebra of functions on $X$ that satisfies the following conditions.

(i) $k$-algebra $A$ is finitely generated

(ii) The evaluation map $e : X \mapsto Mor_{k-alg}(A, k)$ is a bijection. (this map is defined by $e(x)(f) = f(x)$).

The algebra $A$ is called the algebra of polynomial functions on $X$; we usually denote it by $P(X)$.

2. A morphism of affine algebraic varieties $\nu : (X, A) \to (Y, B)$ is a map of sets $\nu : X \to Y$ such that $\nu^*(B) \subset A$.

In other words a morphism $\nu$ is a compatible pair of a map $\nu : X \to Y$ and a morphism of $k$-algebras $\nu^* : B = \mathbf{P}(Y) \to A = P(X)$. Composition of morphisms is defined in a natural way.

With this definition affine algebraic varieties form a category that we denote $Aff_k$.

## 2. Realization.

**Claim.** Let $(X, A)$ be an affine algebraic variety. Choose a collection of elements $y_1, ..., y_n \in A$. Then this collection defines a morphism $\nu : X \to \mathbf{A}^n$. If this collection generates the $k$-algebra $A$ then $\nu$ defines an isomorphism of $X$ with some closed affine subvariety $(Z, P(Z))$ of the affine space $\mathbf{A}^n$.

## 3. Product.

**Claim.** The category $Aff_k$ has products, i.e. given two affine algebraic varieties $X, Y$ there exists an affine algebraic variety $W$ such that for any affine algebraic variety $Z$ we have natural functorial bijection of sets $Mor(Z, W) = Mor(Z, X) \times Mor(Z, Y)$.

This variety, that is defined uniquely up to a canonical isomorphism, is called the product of varieties $X$ and $Y$; standard notation is $X \times Y$.

## 4. Basic open subsets.

Let $(X, A)$ be an affine algebraic variety. Pick any element $finA$. Consider the set $X_f = \{xinX | f(x) \neq 0\}$. We have a natural embedding of algebras $A_f \to F(X_f)$. The pair $(X_f, A_f)$ is an affine algebraic variety.

## 5. Zariski topology.

**Claim.** Let $(X, A)$ be an affine algebraic variety. Collection of all basic open subsets $U \subset X$ is a basis of some topology $\mathcal{T}(X)$. A subset $Z \subset X$ is closed in this topology iff it can be defined by a system of equations $F_\alpha \in A$, i.e. it can be written as $Z = Z(J)$ for some ideal $J \subset A$.

This topology is called the **Zariski topology** on $X$.

## 6. Regular functions.

Let $X$ be an affine algebraic variety, $U \subset X$ and open subset and $f \in F(U)$ a function on $U$.

**Definition**. We say that the function $f$ is **regular at a point** $a \in U$ if there exists a basic open subset $B \subset X$ that is a neighborhood of the point $a$ in $U$ such that the restriction of $f$ to $B$ is a polynomial function on the subset $B$.

We say that the function $f \in F(U)$ is **regular** if it is locally regular, i.e. it is regular at every point $a \in U$.

The algebra of regular functions on the set $U$ we denote by $O(U)$.

**Serre's lemma**. Let $X$ be an affine algebraic variety. Then for every basic open subset $B \subset X$ we have $O(B) = P(B)$, i.e. every regular function is polynomial.

We will prove this lemma later.

**7. Sheaf of functions.** Collection of algebras of regular functions defined on open subsets of an affine algebraic variety is a special case of a general notion.

**Definition**. A sheaf of functions on a topological space $X, \mathcal{T}$ is a collection of $k$-subalgebras $O(U) \subset F(U)$ defined for all open subsets that satisfies the following properties

(i) If $\nu : V \to U$ is an imbedding of open subsets then $\nu^*(O(U)) \subset O(V)$.

(ii) A function $f \in F(U)$ lies in $O(U)$ iff we can cover $U$ by a system of open subsets $V_\alpha$ such that for every index $\alpha$ the restriction of $f$ to the subset $V_\alpha$ lies in $O(V_\alpha)$.

We denote by $O_X$ this sheaf of functions on $X$. Usually we refer to functions $f \in O(U)$ as regular functions.

The conditions above just mean that a function $f \in F(U)$ is regular iff it is locally regular.

**8. Spaces with functions.**

**Definition**. A **space with functions** is a topological space $X, \mathcal{T}$ equipped with a sheaf of functions $O_X$.

A morphism of spaces with functions $\nu : (X, \mathcal{T}(X), O_X) \to (Y, \mathcal{T}(Y), O_Y)$ is a map of sets $\nu : X \to Y$ that is compatible with topology and preserves regularity of functions.

Spaces with functions form a category. As we have seen any affine algebraic variety has canonical structure of a space with functions. Check that if $X, Y$ are affine algebraic varieties then morphisms $\nu : X \to Y$ in the category of spaces with functions are the same as morphisms in the category $Aff_k$.

**9. Algebraic varieties.**

**Definition**. An **algebraic variety** is a space with functions $(X, \mathcal{T}(X), \mathcal{O}(X))$ that satisfies the following condition

(*) There exists a finite covering of $X$ by open subsets $U_i$ such that for every $i$ the space with functions $\mathbf{U}_i$ obtained by restriction of structures to the set $U_i$ is isomorphic to some affine algebraic variety.

Intuitively this means that $X$ is glued from affine algebraic varieties $\mathbf{U}_i$.

Morphisms of algebraic varieties are morphisms of corresponding spaces with functions. Algebraic varieties form a category $Alg_k$.