Math 214-001– Final Exam

Winter 2015

Time: 120 mins.

1. Answer all questions in the spaces provided.
2. Remember to show all work and justify.
3. Except for two notecards of size at most 3x5 inch, no outside assistance is allowed (no calculators).

Name: ______________________________ Section: ____________

Question	Points	Score
1	12	
2	12	
3	8	
4	8	
5	18	
6	14	
7	24	
Total:	96	

1. Consider the matrix $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   (a) (4 points) Is A an orthogonal matrix? Explain.

   (b) (4 points) One of the eigenvalues of A is $\lambda = 1$. What is the geometric multiplicity of λ?.

   (c) (4 points) Does A have other eigenvalues? Explain.

2. Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ be a 6×6 matrix.

(a) (4 points) Compute the reduced row echelon form of A.

(b) (4 points) Give a basis to $\operatorname{Im} A$.

(c) (4 points) Give a basis to $\ker A$.

3. (8 points) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3/5 & 25 \\ 0 & 4/5 & 25 \end{bmatrix}$. Find the QR decomposition of A.

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$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3/5 \\ 4/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 25 \\ 25 \end{bmatrix}$$

4. For which a, b

(a) (2 points) the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & a & b \end{bmatrix}$ is orthogonal.

(b) (2 points) the matrix $B = \begin{bmatrix} 2 & 3 \\ a & 2 \end{bmatrix}$ has real eigenvalues.

(c) (2 points) the matrix $C = \begin{bmatrix} 3 & 3 & 3 \\ 3 & a & 3 \\ b & 3 & 3 \end{bmatrix}$ has an orthonormal eigenbasis.

(d) (2 points) the matrix $D = \begin{bmatrix} b & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & b \end{bmatrix}$ is invertible.

5. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

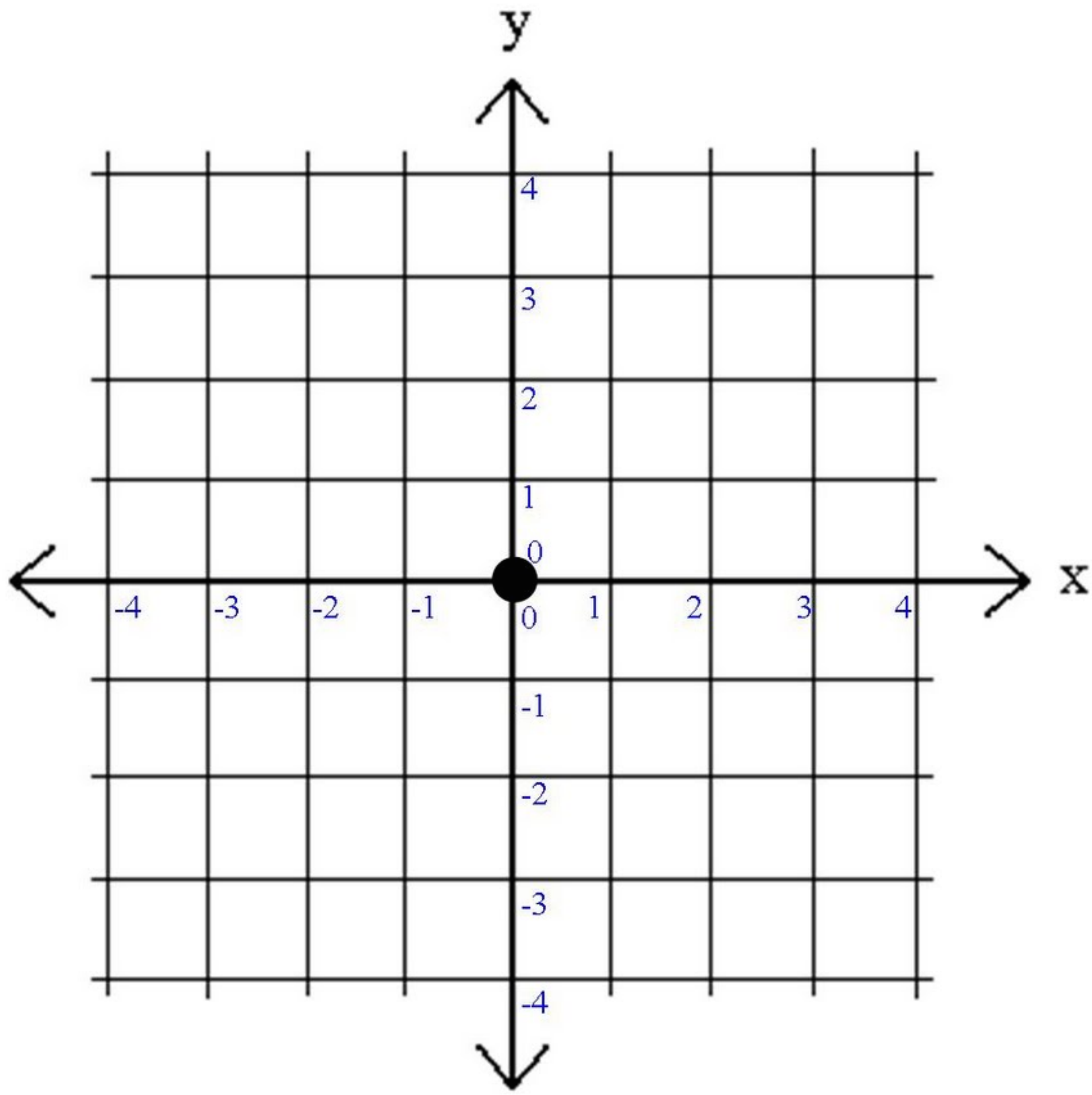
(a) (3 points) Find the eigenvalues of A.

(b) (3 points) Find an eigenbasis for A.

(c) (3 points) Diagonalize A. That is, find matrices S, D such that $S^{-1}AS = D$ is diagonal.

(d) (3 points) Find A^n. You are allowed to write your final answer as a product of at most 3 matrices.

(e) (6 points) Sketch all the points $(x, y) \in \mathbb{R}^2$ satisfying $3x^2 + 2xy + 3y^2 = 1$. ← לא בחומר



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6. The matrix $B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ can be orthogonally diagonalized in the following way

$$B = \begin{bmatrix} -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}^{-1}$$

(there is no need to verify this fact).

(a) (6 points) Find the singular value decomposition $A = U\Sigma V^T$ for the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$.

The image of the unit circle under the map $T(\vec{x}) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \vec{x}$ is an ellipse denoted be $\mathcal{E}$.

(b) (4 points) What are the semi-minor and semi-major of $\mathcal{E}$?

(c) (4 points) What is the area of $\mathcal{E}$? (Hint: the area of the unit circle is π).

7. Determine whether the following statements are true or false, and give justification or a counterexample.

   (a) (4 points) The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is diagonalizable.

   (b) (4 points) Let A be a real 6×6 matrix with eigenvalue $1 + i$ of algebraic multiplicity 3. Then $trace(A) = 6$.

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   (c) (4 points) For any 8×8-matrix A, $\det(-A) = \det(A)$.

(d) (4 points) Let v_1 and v_2 be vectors in $\mathbb{R}^3$. The determinant of the 3×3 matrix with columns v_1, v_2, and $2v_1 + 3v_2$ is 0.

(e) (4 points) A 5×5 real matrix has a real eigenvalue.

(f) (4 points) The matrices $A = \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ are similar.