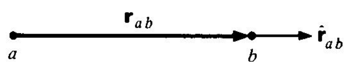
known geometry. The first measurements were performed by Henry Cavendish in 1771 using a torsion balance. The modern value of G is 6.67×10^{-11} N·m²/kg². (G is the least accurately known of the fundamental constants. Perhaps you can devise a new way to measure it more precisely.) Experimentally, G is the same for all materials—aluminum, lead, neutrons, or what have you. For this reason, the law is called the universal law of gravitation.



The gravitational force between two particles is *central* (along the line of centers) and attractive. The simplest way to describe these properties is to use vectors. By convention, we introduce a vector $\mathbf{r}_{ab}$ *from* the particle exerting the force, particle a in this case, *to* the particle experiencing the force, particle b. Note that $|\mathbf{r}_{ab}| = r$. Using the unit vector $\hat{\mathbf{r}}_{ab} = \mathbf{r}_{ab}/r$, we have

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab}.$$

The negative sign indicates that the force is attractive. The force on a due to b is

$$\mathbf{F}_a = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ba} = +\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab} = -\mathbf{F}_b,$$

since $\hat{\mathbf{r}}_{ba} = -\hat{\mathbf{r}}_{ab}$. The forces are equal and opposite, and Newton's third law is automatically satisfied.

The gravitational force has a unique and mysterious property. Consider the equation of motion of particle b under the gravitational attraction of particle a.

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab}$$

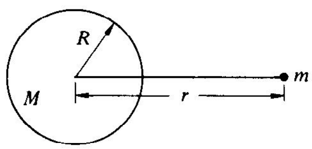
$$= M_b\mathbf{a}_b$$

or

$$\mathbf{a}_b = -\frac{GM_a}{r^2}\hat{\mathbf{r}}_{ab}.$$

The acceleration of a particle under gravity is independent of its mass! There is a subtle point connected with our cancelation of M_b, however. The "mass" (*gravitational* mass) in the law of gravitation, which measures the strength of gravitational interaction, is operationally distinct from the "mass" (*inertial* mass) which characterizes inertia in Newton's second law. Why gravitational mass is proportional to inertial mass for all matter is one of the great mysteries of physics. However, the proportionality has been experimentally verified to very high accuracy, approximately 1 part in 10^{11}; we shall have more to say about this in Chap. 8.

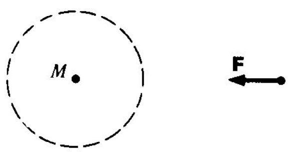
The Gravitational Force of a Sphere The law of gravitation applies only to particles. How can we find the gravitational force on a particle due to an extended body like the earth? Fortunately, the gravitational force obeys the *law of superposition:* the force due to a collection of particles is the vector sum of the forces exerted by the particles individually. This allows us to mentally divide the body into a collection of small elements which can be treated as particles. Using integral calculus, we can sum the forces from all the particles. This method is applied in Note 2.1 to calculate the force between a particle of mass m and a uniform thin spherical shell of mass M and radius R. The result is



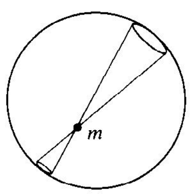
$$\mathbf{F} = -G\frac{Mm}{r^2}\hat{\mathbf{r}} \qquad r > R$$

$$\mathbf{F} = 0 \qquad r < R,$$

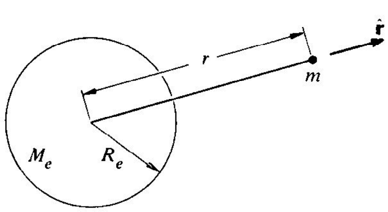
where r is the distance from the center of the shell to the particle. If the particle lies outside the shell, the force is the same as if all the mass of the shell were concentrated at its center.



The reason the gravitational force vanishes inside the spherical shell can be seen by a simple argument due to Newton. Consider the two small mass elements marked out by a conical surface with its apex at m. The amount of mass in each element is proportional to its surface area. The area increases as (distance)^2. However, the strength of the force varies as 1/(distance)^2. Thus the forces of the two mass elements are equal and opposite, and cancel. The total force on m is zero, because we can pair up all the elements of the shell this way.



A uniform solid sphere can be regarded as a succession of thin spherical shells, and it follows that for particles outside it, a sphere behaves gravitationally as if its mass were concentrated at its center. This result also holds if the density of the sphere varies with radius, provided the mass distribution is spherically symmetric. For example, although the earth has a dense core, the mass distribution is nearly spherically symmetric, so that to good approximation the gravitational force of the earth on a mass m at distance r is



$$\mathbf{F} = -\frac{GM_em}{r^2}\hat{\mathbf{r}} \qquad r \geq R_e,$$

where M_e is the mass of the earth and R_e is its radius.

At the surface of the earth, the gravitational force is

$$\mathbf{F} = -\frac{GM_e m}{R_e^2}\hat{\mathbf{r}},$$

and the acceleration due to gravity is

$$\mathbf{a} = \frac{\mathbf{F}}{m} = -\frac{GM_e}{R_e^2}\hat{\mathbf{r}}.$$

As we expect, the acceleration is independent of m. GM_e/R_e^2 is usually called g. Sometimes g is written as a vector directed down. toward the center of the earth.

$$\mathbf{g} = -\frac{GM_e}{R_e^2}\hat{\mathbf{r}}$$

Numerically, $|g|$ is approximately 9.8 m/s^2 = 980 cm/s^2 $\approx$ 32 ft/s^2.

By convention, g usually stands for the downward acceleration of an object measured with respect to the earth's surface. This differs slightly from the true gravitational acceleration because of the rotation of the earth, a point we shall return to in Chap. 8. g increases by about five parts per thousand from the equator to the poles. About half this variation is due to the slight flattening of the earth about the poles, and the remainder arises from the earth's rotation. Local mass concentrations also affect g; a variation in g of ten parts per million is typical.

The acceleration due to gravity decreases with altitude. We can estimate this effect by taking differentials of the expression

$$g(r) = \frac{GM_e}{r^2}.$$

We have

$$\Delta g(r) = \frac{dg}{dr}\Delta r = -\frac{2GM_e}{r^3}\Delta r = -\frac{2g}{r}\Delta r.$$

The fractional change in g with altitude is

$$\frac{\Delta g}{g} = -\frac{2\,\Delta r}{r}.$$

At the earth's surface, $r = 6 \times 10^6$ m, and g decreases by one part per million for an increase in altitude of 3 m.

Weight We define the weight of a body near the earth to be the gravitational force exerted on it by the earth. At the surface of the earth the weight of a mass m is

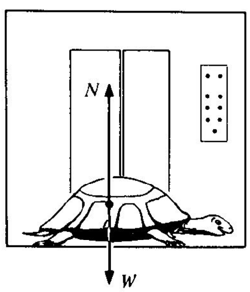
$$\mathbf{W} = -G\frac{M_e m}{R_e^2}\hat{\mathbf{r}}$$
$$= m\mathbf{g}.$$

The unit of weight is the newton (SI), dyne (cgs), or pound (English). Since $g = 9.8$ m/s², the weight of 1 kg mass is 9.8 N. An automobile which weighs 3,200 lb has mass

$$m = \frac{W}{g} = \frac{3{,}200 \text{ lb}}{32 \text{ ft/s}^2} = 100 \text{ slugs}.$$

Our definition of weight is unambiguous. According to our definition, the weight of a body is not affected by its motion. However, weight is often used in another sense. In this sense, the magnitude of the weight is the magnitude of the force which must be exerted on a body by its surroundings to keep it at rest; its direction is the direction of gravitational attraction. The next example illustrates the difference between these two definitions.

Example 2.9 **Turtle in an Elevator**



An amiable turtle of mass M stands in an elevator accelerating at rate a Find N, the force exerted on him by the floor of the elevator.

The forces acting on the turtle are N and the weight, the true gravitational force $W = Mg$. Taking up to be the positive direction, we have

$$N - W = Ma$$
$$N = Mg + Ma$$
$$= M(g + a).$$

This result illustrates the two senses in which weight is used. In the sense that weight is the gravitational force, the weight of the turtle, Mg, is independent of the motion of the elevator. In contrast, the weight of the turtle has magnitude $N = M(g + a)$, if the magnitude of the weight is taken to be the magnitude of the force exerted by the elevator on the turtle. If the turtle were standing on a scale, the scale would indicate a weight N. With this definition, the turtle's weight increases when the elevator accelerates up.

If the elevator accelerates down, a is negative and N is less than Mg. If the downward acceleration equals g, N becomes zero, and the turtle "floats" in the elevator. The turtle is then said to be in a state of weightlessness.

Although the two definitions of weight are both commonly used and are both acceptable, we shall generally consider weight to mean the true gravitational force. This is consistent with our resolve to refer all motion to inertial systems and helps us to keep the real forces on a body distinct. If the acceleration due to gravity is g, the real gravitational force on a body of mass m is $W = mg$.

Our definition of weight has one minor drawback. As we saw in the last example, a scale does not read mg in an accelerating system. As we have already pointed out, systems at rest on the earth's surface have a small acceleration due to the earth's rotation, so that the reading of a scale is not the true gravitational force on a mass. However, the effect is small, and we shall treat the surface of the earth as an inertial system for the present.

The Gravitational Field The gravitational force on particle b due to particle a is

$$\mathbf{F}_b = -\frac{GM_aM_b}{r^2}\hat{\mathbf{r}}_{ab},$$

where $\hat{\mathbf{r}}_{ab}$ is a unit vector which points from a toward b. The ratio $\mathbf{F}_b/M_b$, which is independent of M_b, is called the *gravitational field* due to M_a. Denoting the field by $\boldsymbol{G}_a$, we have

$$\begin{aligned}\boldsymbol{G}_a &= \frac{\mathbf{F}_b}{M_b}\\ &= -G\frac{M_a}{r^2}\hat{\mathbf{r}}_{ab}.\end{aligned}$$

In general, if the gravitational field at a point in space is $\boldsymbol{G}$, the gravitational force on mass M at that point is

$$\mathbf{F} = M\boldsymbol{G}.$$

The dimension of gravitation field is force/mass = acceleration. The acceleration of mass M by gravitational field $\boldsymbol{G}$ is given by

$$\begin{aligned}\mathbf{F} &= M\mathbf{a}\\ &= M\boldsymbol{G}\end{aligned}$$

or

$$\mathbf{a} = \boldsymbol{G}.$$

We see that the gravitational field at a point is numerically equal to the gravitational acceleration experienced by a body located there. For example, the gravitational field of the earth is **g**.

For the present we can regard the gravitational field as a mathematical convenience that allows us to focus on the source of the gravitational attraction. However, the concept of field has a broader significance in physics. Fields have important physical properties, such as the ability to store or transmit energy and momentum. Until recently, the dynamical properties of the gravitational field were chiefly of theoretical interest, since their effects were too small to be observed. However, there is now lively experimental activity in searching for such dynamical features as gravitational waves and "black holes."

The Electrostatic Force

We mention the electrostatic force only in passing since its full implications are better left to a more detailed study of electricity and magnetism. The salient feature of the electrostatic force between two particles is that the force, like gravity, is an inverse square central force. The force depends upon a fundamental property of the particle called its *electric charge* q. There are two different kinds of electric charge: like charges repel, unlike charges attract.

For the sake of convenience, we distinguish the two different kinds of charges by associating an algebraic sign with q, and for this reason we talk about negative and positive charges. The electrostatic force F_b on charge q_b due to charge q_a is given by Coulomb's law:

$$\mathbf{F}_b = k\frac{q_a q_b}{r^2}\hat{\mathbf{r}}_{ab}.$$

k is a constant of proportionality and $\hat{\mathbf{r}}_{ab}$ is a unit vector which points from a to b. If q_a and q_b are both negative or both positive, the force is repulsive, but if the charges are of different sign, $\mathbf{F}_b$ is attractive.

In the SI system, the unit of charge is the *coulomb*, abbreviated C. (The coulomb is defined in terms of electric currents and magnetic forces.) In this system, k is found by experiment to be

$$k = 8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2.$$

In analogy with the gravitational field, we can define the electric field **E** as the electric force on a body divided by its charge. The electric field at **r** due to a charge q at the origin is

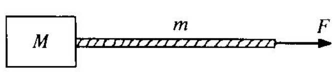
$$\mathbf{E} = k\frac{q}{r^2}\hat{\mathbf{r}}.$$

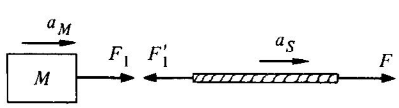
Contact Forces

By contact forces we mean the forces which are transmitted between bodies by short-range atomic or molecular interactions. Examples include the pull of a string, the surface force of sliding friction, and the force of viscosity between a moving body and a fluid. One of the achievements of twentieth century physics is that these forces can now be explained in terms of the fundamental properties of matter. However, our approach will emphasize the empirical properties of these forces and the techniques for dealing with them in physical problems, with only brief mention of their microscopic origins.

Tension—The Force of a String We have been taking the "string" force for granted, having some primitive idea of this kind of force. The following example is intended to help put these ideas into sharper focus.

Example 2.10 **Block and String 3**





Consider a block of mass M in free space pulled by a string of mass m. A force F is applied to the string, as shown. What is the force that the string "transmits" to the block?

The sketch shows the force diagrams. F_1 is the force of the string on the block, F_1' is the force of the block on the string, a_M is the acceleration of the block, and a_S is the acceleration of the string. The equations of motion are

$$F_1 = Ma_M$$

$$F - F_1' = ma_S.$$

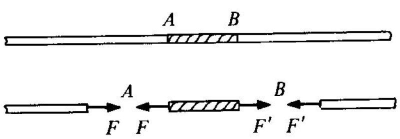
Assuming that the string is inextensible, it accelerates at the same rate as the block, giving the constraint equation $a_S = a_M$. Furthermore, $F_1 = F_1'$ by Newton's third law. Solving for the acceleration, we find that

$$a = \frac{F}{M + m},$$

as we expect, and

$$F_1 = F_1'$$
$$= \frac{M}{M+m} F.$$

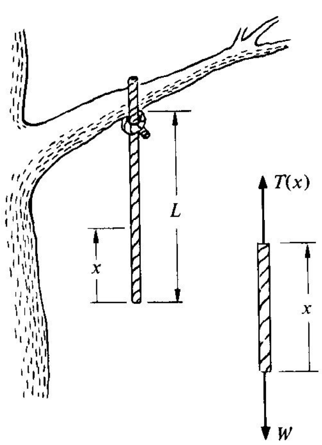
The force on the block is less than F; the string does not transmit the full applied force. However, if the mass of the string is negligible compared with the block, $F_1 = F$ to good approximation.



We can think of a string as composed of short sections interacting by contact forces. Each section pulls the sections to either side of it, and by Newton's third law, it is pulled by the adjacent sections. The magnitude of the force acting between adjacent sections is called *tension*. There is no direction associated with tension. In the sketch, the tension at A is F and the tension at B is F'.

Although a string may be under considerable tension (for example a string on a guitar), if the tension is uniform, the net string force on each small section is zero and the section remains at rest unless external forces act on it. If there are external forces on the section, or if the string is accelerating, the tension generally varies along the string, as Examples 2.11 and 2.12 show.

Example 2.11 **Dangling Rope**



A uniform rope of mass M and length L hangs from the limb of a tree. Find the tension a distance x from the bottom.

The force diagram for the lower section of the rope is shown in the sketch. The section is pulled up by a force of magnitude $T(x)$, where $T(x)$ is the tension at x. The downward force on the rope is its weight $W = Mg(x/L)$. The total force on the section is zero since it is at rest. Hence

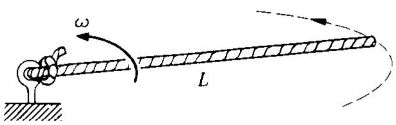
$$T(x) = \frac{Mg}{L} x.$$

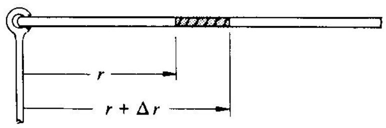
At the bottom of the rope the tension is zero, while at the top the tension equals the total weight of the rope Mg.

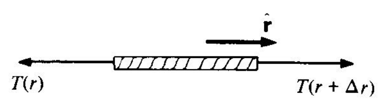
The next example cannot be solved by direct application of Newton's second law. However, by treating each small section of the system as a particle, and taking the limit using calculus, we can obtain a differential equation which leads to the solution.

The technique is so useful that it is employed time and again in physics.

Example 2.12 **Whirling Rope**







A uniform rope of mass M and length L is pivoted at one end and whirls with uniform angular velocity ω. What is the tension in the rope at distance r from the pivot? Neglect gravity.

Consider the small section of rope between r and $r + \Delta r$. The length of the section is Δr and its mass is $\Delta m = M\,\Delta r/L$. Because of its circular motion, the section has a radial acceleration. Therefore, the forces pulling either end of the section cannot be equal, and we conclude that the tension must vary with r.

The inward force on the section is $T(r)$, the tension at r, and the outward force is $T(r + \Delta r)$. Treating the section as a particle, its inward radial acceleration is $r\omega^2$. [This point can be confusing; it is just as reasonable to take the acceleration to be $(r + \Delta r)\omega^2$. However, we shall shortly take the limit $\Delta r \to 0$, and in this limit the two expressions give the same result.]

The equation of motion for the section is

$$T(r + \Delta r) - T(r) = -(\Delta m)r\omega^2 = -\frac{Mr\omega^2\,\Delta r}{L}.$$

The problem is to find $T(r)$, but we are not yet ready to do this. However, by dividing the last equation by Δr and taking the limit $\Delta r \to 0$, we can find an exact expression for dT/dr.

$$\frac{dT}{dr} = \lim_{\Delta r \to 0} \frac{T(r + \Delta r) - T(r)}{\Delta r} = -\frac{Mr\omega^2}{L}$$

To find the tension, we integrate.

$$dT = -\frac{M\omega^2}{L}\, r\, dr$$

$$\int_{T_0}^{T(r)} dT = -\int_0^r \frac{M\omega^2}{L}\, r\, dr,$$

where T_0 is the tension at $r = 0$.

$$T(r) - T_0 = -\frac{M\omega^2}{L}\frac{r^2}{2}$$

or

$$T(r) = T_0 - \frac{M\omega^2}{2L} r^2.$$

To evaluate T_0 we need one additional piece of information. Since the end of the rope at $r = L$ is free, the tension there must be zero. We have

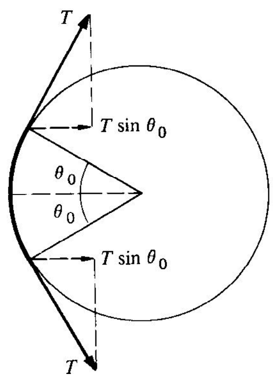
$$T(L) = 0 = T_0 - \tfrac{1}{2}M\omega^2 L.$$

Hence, $T_0 = \frac{1}{2}M\omega^2 L$, and the final result can be written

$$T(r) = \frac{M\omega^2}{2L}(L^2 - r^2).$$

When a pulley is used to change the direction of a rope under tension, there is a reaction force on the pulley. As every sailor knows, the force on the pulley depends on the tension and the angle through which the rope is deflected. Working out this problem in detail provides another illustration of how calculus can be applied to a physical problem.

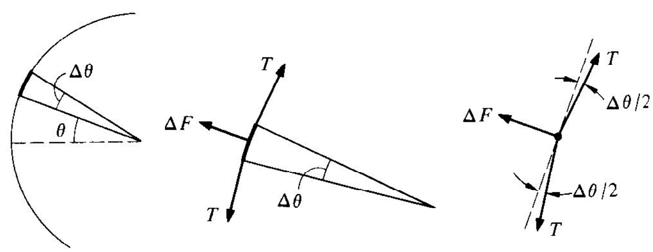
Example 2.13 **Pulleys**



A string with constant tension T is deflected through angle $2\theta_0$ by a smooth fixed pulley. What is the force on the pulley?

Intuitively, the magnitude of the force is $2T \sin \theta_0$. To prove this result, we shall find the force due to each element of the string and then add them vectorially.

Consider the section of string between θ and $\theta + \Delta\theta$. The force diagram is drawn below, center. ΔF is the outward force due to the pulley

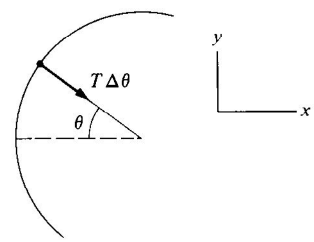


The tension in the string is constant, but the forces T at either end of the element are not parallel. Since we shall shortly take the limit $\Delta\theta \to 0$, we can treat the element like a particle. For equilibrium, the total force is zero. We have

$$\Delta F - 2T \sin \frac{\Delta\theta}{2} = 0.$$

For small $\Delta\theta$, $\sin(\Delta\theta/2) \approx \Delta\theta/2$ and

$$\Delta F = 2T\frac{\Delta\theta}{2} = T\Delta\theta.$$

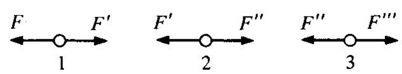


Thus the element exerts an inward radial force of magnitude $T\,\Delta\theta$ on the pulley.

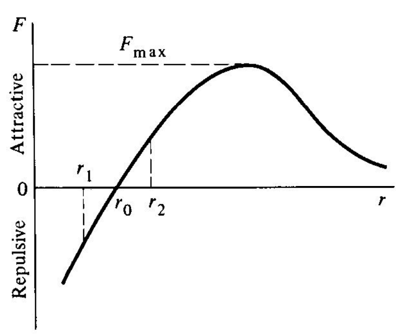
The element at angle θ exerts a force in the x direction of $(T\,\Delta\theta)\cos\theta$. The total force in the x direction is $\Sigma T\cos\theta\,\Delta\theta$, where the sum is over all elements of the string which are touching the pulley. In the limit $\Delta\theta \to 0$, the sum becomes an integral. The total force in the x direction is therefore

$$\int_{-\theta_0}^{\theta_0} T\cos\theta\,d\theta = 2T\sin\theta_0.$$

Tension and Atomic Forces The force on each element of a string in equilibrium is zero. Nevertheless, the string will break if the tension is too large. We can understand this qualitatively by looking at strings from the atomic viewpoint. An idealized model of a string is a single long chain of molecules. Suppose that force F is applied to molecule 1 at the end of the string. The force diagrams for molecules 1 and 2 are shown in the sketch below. In



equilibrium, $F = F'$ and $F' = F''$, so that $F'' = F$. We see that the string "transmits" the force F. To understand how this comes about, we need to look at the nature of intermolecular forces.



Qualitatively, the force between two molecules depends on the distance r between them, as shown in the drawing. The intermolecular force is repulsive at small distances, is zero at some separation r_0, and is attractive for $r > r_0$. For large values of r the force falls to zero. There are no scales on our sketch, but r_0 is typically a few angstroms (1 Å $= 10^{-10}$ m).

When there is no applied force, the molecules must be a distance r_0 apart; otherwise the intermolecular forces would make the string contract or expand. As we pull on the string, the molecules move apart slightly, say to $r = r_2$, where the intermolecular attractive force just balances the applied force so that the total force on each molecule is zero. If the string were stiff like a metal rod, we could push as well as pull. A push makes the molecules move slightly together, say to $r = r_1$, where the intermolecular repulsive force balances the applied force. The change in the length depends on the slope of the interatomic force curve at r_0. The steeper the curve, the less the stretch for a given pull.

The attractive intermolecular force has a maximum value F_{max}, as shown in the sketch. If the applied pull is greater than F_{max}, the intermolecular force is too weak to restore balance—the molecules continue to separate and the string breaks.

For a real string or rod, the intermolecular forces act in a three dimensional lattice work of atoms. The breaking strength of most materials is considerably less than the limit set by F_{max}. Breaks occur at points of weakness, or "defects," in the lattice, where the molecular arrangement departs from regularity. Microscopic metal "whiskers" seem to be nearly free from defects, and they exhibit breaking strengths close to the theoretical maximum.

The Normal Force The force exerted by a surface on a body in contact with it can be resolved into two components, one perpendicular to the surface and one tangential to the surface. The perpendicular component is called the *normal* force and the tangential component is called *friction.*

The origin of the normal force is similar to the origin of tension in a string. When we put a book on a table, the molecules of the book exert downward forces on the molecules of the table. The molecules composing the upper layers of the tabletop move downward until the repulsion of the molecules below balances the force applied by the book. From the atomic point of view, no surface is perfectly rigid. Although compression always occurs, it is often too slight to notice, and we shall neglect it and treat surfaces as rigid.

The normal force on a body, generally denoted by N, has the following simple property: for a body resting on a surface, N is equal and opposite to the resultant of all other forces which act on the body in a direction perpendicular to the surface. For instance, when you stand still, the normal force exerted by the ground is equal to your weight. However, when you walk, the normal force fluctuates as you accelerate up and down.

Friction Friction cannot be described by a simple formula, but macroscopic mechanics is hard to understand without some idea of the properties of friction.

Friction arises when the surface of one body moves, or tries to move, along the surface of a second body. The magnitude of the force of friction varies in a complicated way with the nature of the surfaces and their relative velocity. In fact, the only thing we can always say about friction is that it opposes the motion which would occur in its absence. For instance, suppose that we try to push a book across a table. If we push gently, the book remains at rest; the force of friction assumes a value equal and opposite to the tangential force we apply. In this case, the force of friction assumes whatever value is needed to keep the book at rest. However, the friction force cannot increase indefinitely. If we push hard enough, the book starts to slide. For many surfaces the maximum value of the friction is found to be equal to μN, where N is the normal force and μ is the *coefficient of friction.*

When a body slides across a surface, the friction force is directed opposite to the instantaneous velocity and has magnitude μN. Experimentally, the force of sliding friction decreases slightly when bodies begin to slide, but for the most part we shall neglect this effect. For two given surfaces the force of sliding friction is essentially independent of the area of contact.

It may seem strange that friction is independent of the area of contact. The reason is that the actual area of contact on an atomic scale is a minute fraction of the total surface area. Friction occurs because of the interatomic forces at these minute regions of atomic contact. The fraction of the geometric area in atomic contact is proportional to the normal force divided by the geometric area. If the normal force is doubled, the area of atomic contact is doubled and the friction force is twice as large. However, if the geometric area is doubled while the normal force remains the same, the fraction of area in atomic contact is halved and the actual area in atomic contact—hence the friction force—remains constant. (Nonrigid bodies, like automobile tires, are more complicated. A wide tire is generally better than a narrow one for good acceleration and braking.)

In summary, we take the force of friction f to behave as follows:

1. For bodies not in relative motion,

$$0 \leq f \leq \mu N.$$

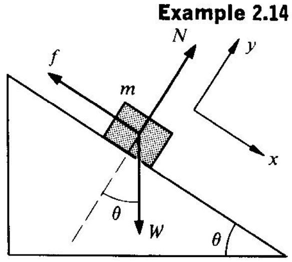
f opposes the motion that would occur in its absence.

2. For bodies in relative motion,

$$f = \mu N.$$

f is directed opposite to the relative velocity.

Example 2.14 **Block and Wedge with Friction**



A block of mass m rests on a fixed wedge of angle θ. The coefficient of friction is μ. (For wooden blocks, μ is of the order of 0.2 to 0.5.) Find the value of θ at which the block starts to slide.

In the absence of friction, the block would slide down the plane; hence the friction force f points up the plane. With the coordinates shown, we have

$$m\ddot{x} = W \sin\theta - f$$

and

$$m\ddot{y} = N - W\cos\theta$$
$$= 0.$$

When sliding starts, f has its maximum value μN, and $\ddot{x} = 0$. The equations then give

$$W\sin\theta_{\max} = \mu N$$
$$W\cos\theta_{\max} = N.$$

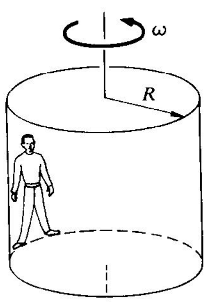
Hence,

$$\tan\theta_{\max} = \mu.$$

Notice that as the wedge angle is gradually increased from zero, the friction force grows in magnitude from zero toward its maximum value μN, since before the block begins to slide we have

$$f = W\sin\theta \qquad \theta \le \theta_{\max}.$$

Example 2.15 **The Spinning Terror**

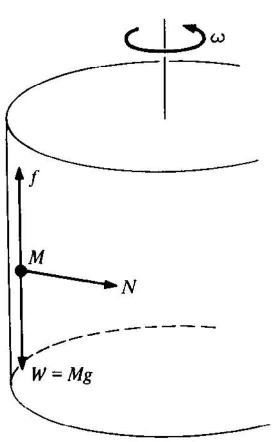


The Spinning Terror is an amusement park ride—a large vertical drum which spins so fast that everyone inside stays pinned against the wall when the floor drops away. What is the minimum steady angular velocity ω which allows the floor to be dropped away safely?

Suppose that the radius of the drum is R and the mass of the body is M. Let μ be the coefficient of friction between the drum and M. The forces on M are the weight W, the friction force f, and the normal force exerted by the wall, N, as shown below.

The radial acceleration is $R\omega^2$ toward the axis, and the radial equation of motion is

$$N = MR\omega^2.$$



By the law of static friction,

$$f \le \mu N = \mu MR\omega^2.$$

Since we require M to be in vertical equilibrium,

$$f = Mg,$$

and we have

$$Mg \le \mu MR\omega^2$$

or

$$\omega^2 \ge \frac{g}{\mu R}.$$

The smallest value of ω that will work is

$$\omega_{\min} = \sqrt{\frac{g}{\mu R}}.$$

For cloth on wood μ is at least 0.3, and if the drum has radius 6 ft, then $\omega_{\min} = [32/(0.3 \times 6)]^{\frac{1}{2}} = 4$ rad/s. The drum must make at least $\omega/2\pi = 0.6$ turns per second.

Viscosity

A body moving through a liquid or gas is retarded by the force of viscosity exerted on it by the fluid. Unlike the friction force between dry surfaces, the viscous force has a simple velocity dependence; it is proportional to the velocity. At high speeds other forces due to turbulence occur and the total drag force can have a complicated velocity dependence. (Sports car designers use a force proportional to the square of the speed to account for the drag forces.) However, in many practical cases viscosity is the only important drag force.

Viscosity arises because a body moving through a medium exerts forces which set the nearby fluid into motion. By Newton's third law the fluid exerts a reaction force on the body.

We can write the viscous retarding force in the form

$$\mathbf{F}_v = -C\mathbf{v},$$

where C is a constant which depends on the fluid and the geometry of the body. $\mathbf{F}_v$ is always along the line of motion, because it is proportional to $\mathbf{v}$. The negative sign assures that $\mathbf{F}_v$ opposes the motion. For objects of simple shape moving through a gas at low pressure, C can be calculated from first principles. We shall treat it as an empirical constant.

When the only force on a body is the viscous retarding force, the equation of motion is

$$-C\mathbf{v} = m\frac{d\mathbf{v}}{dt}.$$

What we have here is a differential equation for $\mathbf{v}$. Since the force is along the line of motion, only the magnitude of $\mathbf{v}$ changes[1]

[1] Formally, this is proved as follows. Since $\mathbf{v} = v\hat{\mathbf{v}}$, $d\mathbf{v}/dt = dv/dt\,\hat{\mathbf{v}} + v\,d\hat{\mathbf{v}}/dt$. The equation of motion is $-Cv\hat{\mathbf{v}} = m\,dv/dt\,\hat{\mathbf{v}} + mv\,d\hat{\mathbf{v}}/dt$. Because $\hat{\mathbf{v}}$ is a unit vector, $d\hat{\mathbf{v}}/dt$ is perpendicular to $\hat{\mathbf{v}}$. The other terms of the equation lie in the $\hat{\mathbf{v}}$ direction, so that $d\hat{\mathbf{v}}/dt$ must be zero. The same conclusion follows more directly from the simple physical argument that a force directed along the line of motion can change the speed but cannot change the direction of motion.

and the vector equation reduces to the scalar equation

$$-Cv = m\frac{dv}{dt}$$

or

$$m\frac{dv}{dt} + Cv = 0.$$

The task of solving such a differential equation occurs often in physics. A few differential equations are so simple and occur so frequently that it is helpful to be thoroughly familiar with them and their solutions. The equation of the form $m\,dv/dt + Cv = 0$ is one of the most common, and the following example should make you feel at home with it.

Example 2.16 Free Motion in a Viscous Medium

A body of mass m released with velocity v_0 in a viscous fluid is retarded by a force Cv. Find the motion, supposing that no other forces act.

The equation of motion is

$$m\frac{dv}{dt} + Cv = 0,$$

which we can rewrite in the standard form

$$\frac{dv}{dt} + \frac{C}{m}v = 0. \qquad 1$$

If you are familiar with the properties of the exponential function e^{ax}, then you know that $(d/dx)e^{ax} = ae^{ax}$, or $(d/dx)e^{ax} - ae^{ax} = 0$. This suggests that we use a trial solution $v = e^{at}$, where a is a constant to be determined. Then $dv/dt = ae^{at}$, and substituting this in Eq. (1) gives us

$$ae^{at} + \frac{C}{m}e^{at} = 0.$$

This holds true at all times if $a = -C/m$. Hence, a solution is

$$v = e^{-Ct/m}.$$

However, this cannot be the correct solution; v has the dimension of velocity whereas the exponential function is dimensionless. Let us try

$$v = Ae^{-Ct/m},$$

where A is a constant. Substituting this in Eq. (1) gives

$$-\frac{C}{m}Ae^{-Ct/m} + \frac{C}{m}Ae^{-Ct/m} = 0,$$

so that the solution is acceptable. But A can be any constant, whereas our solution must be quite specific. To evaluate A we make use of the given initial condition. An *initial condition* is a specific piece of information about the motion at some particular time. We were given that $v = v_0$ at $t = 0$. Hence

$$v(t = 0) = Ae^0 = v_0.$$

Since $e^0 = 1$, it follows that $A = v_0$, and the full solution is

$$v = v_0 e^{-Ct/m}.$$

We solved Eq. (1) by what might be called a common sense approach—we simply guessed the answer. This particular equation can also be solved by formal integration after appropriate "separation of the variables."

$$\frac{dv}{dt} + \frac{C}{m}v = 0$$

$$\frac{dv}{v} = -\frac{C}{m}dt$$

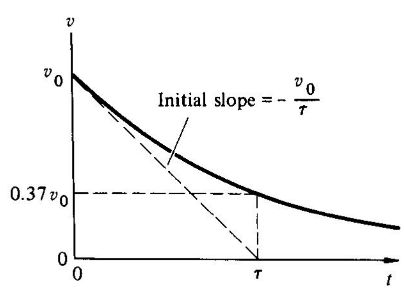
$$\int_{v_0}^{v} \frac{dv}{v} = -\int_0^t \frac{C}{m}dt$$

Note the correspondence between the limits: v is the velocity at time t and v_0 is the velocity at time 0.

$$\ln\frac{v}{v_0} = -\frac{C}{m}t$$

$$\frac{v}{v_0} = e^{(-C/m)t}$$

$$v = v_0 e^{-Ct/m}.$$



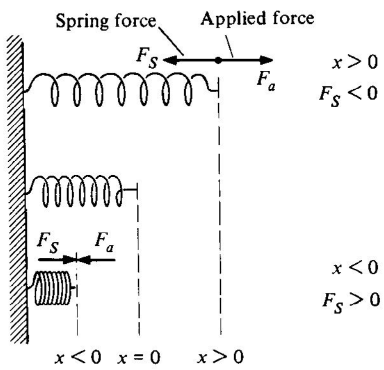
Before leaving this problem, let us look at the solution in a little more detail. The velocity decreases exponentially in time. If we let $\tau = m/c$, then we have $v = v_0 e^{-t/\tau}$. τ is a *characteristic time* for the system; it is the time for the velocity to drop to $e^{-1} \approx 0.37$ of its original velocity.

The Linear Restoring Force: Hooke's Law, the Spring, and Simple Harmonic Motion

In the mid-seventeenth century Robert Hooke discovered that the extension of a spring is proportional to the applied force, both for positive and negative displacements. The force F_S exerted by a stretched spring is given by Hooke's law

$$F_S = -kx,$$

where k is a constant called the *spring constant* and x is the displacement of the end of the spring from its equilibrium position. The magnitude of F_S increases linearly with displacement. The



negative sign indicates that F_S is a restoring force; the spring force is always in the direction that tends to restore the spring to its equilibrium length. A force obeying Hooke's law is called a *linear restoring force.*

If the spring is stretched by an applied force F_a, then $x > 0$ and F_S is negative, directed toward the origin.

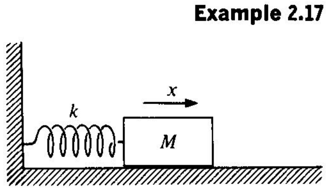
If the spring is compressed by F_a, then $x < 0$ and F_S is positive.

Hooke's law is essentially empirical and breaks down for large displacements. Taking a jaundiced view of affairs, we could rephrase Hooke's law as "extension is proportional to force, as long as it is." However, this misses the important point. For sufficiently small displacements Hooke's law is remarkably accurate, not only for springs but also for practically every system near equilibrium. Consequently, the motion of a system under a linear restoring force occurs persistently throughout physics.

By looking at the intermolecular force curve on page 91, we can see why the linear restoring force is so common. If the force curve is linear in the neighborhood of the equilibrium point, then the force is proportional to the displacement from equilibrium. This is almost always the case; a sufficiently short segment of a curve is generally linear to good approximation. Only in pathological cases does the force curve have no linear component. It is also apparent that the linear approximation necessarily breaks down for large displacements. We shall return to these considerations in Chap. 4.

In the following example we investigate simple harmonic motion—the motion of a mass under a linear restoring force. We shall again encounter a differential equation. Like the equation for viscous drag, the differential equation for simple harmonic motion occurs frequently and is well worth learning to recognize early in the game. Fortunately, the solution has a simple form.

Example 2.17 **Spring and Block—The Equation for Simple Harmonic Motion**



A block of mass M is attached to one end of a horizontal spring, the other end of which is fixed. The block rests on a horizontal frictionless surface. What motion is possible for the block?

Since the spring force is the only horizontal force acting on the block, the equation of motion is

$$M\ddot{x} = -kx$$

or

$$\ddot{x} + \frac{k}{M}x = 0,$$

where x is measured from the equilibrium position. It is convenient to write

$$\omega = \sqrt{\frac{k}{M}}.$$

The equation takes the standard form

$$\ddot{x} + \omega^2 x = 0.$$

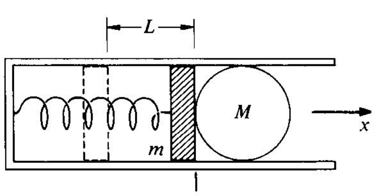
You should learn to recognize the mathematical form of this equation, since it arises in many different physical contexts. It is called the equation of *simple harmonic motion* (SHM). Without going into the theory of differential equations, we simply write down the solution

$$x = A \sin \omega t + B \cos \omega t.$$

ω is known as the *angular frequency* of the motion. By substitution it is easy to show that this solution satisfies the original equation for arbitrary values of A and B. The theory of differential equations tells us that there are no further nontrivial solutions. The main point here, however, is to become familiar with the mathematical form of the SHM differential equation and the form of its solution. We shall derive the solution in Example 4.2, but this purely mathematical process does not concern us now.

As we show in the following example, the constants A and B are to be determined from the initial conditions. We shall show that A and B can be found by knowing the position and velocity at some particular time.

Example 2.18 The Spring Gun—An Example Illustrating Initial Conditions



The piston of a spring gun has mass m and is attached to one end of a spring with spring constant k. The projectile is a marble of mass M. The piston and marble are pulled back a distance L from the equilibrium position and suddenly released. What is the speed of the marble as it loses contact with the piston? Neglect friction.

Let the x axis be along the direction of motion with the origin at the unstretched position. The position of the piston is given by

$$x(t) = A \sin \omega t + B \cos \omega t, \qquad 1$$

where $\omega = \sqrt{k/(m + M)}$. This equation holds up to the time the marble and piston lose contact. The velocity is

$$v(t) = \dot{x}(t)$$
$$= \omega A \cos \omega t - \omega B \sin \omega t. \qquad 2$$

There are two arbitrary constants in the solution, A and B, and to evaluate them we need two pieces of information. We know that at $t = 0$, when the spring is released, the position and velocity are given by

$$x(0) = -L$$
$$v(0) = 0.$$

Using these values in Eqs. (1) and (2), we find

$$\begin{aligned} -L &= x(0) \\ &= A \sin(0) + B \cos(0) \\ &= B, \end{aligned}$$

and

$$\begin{aligned} 0 &= v(0) \\ &= \omega A \cos(0) - \omega B \sin(0) \\ &= \omega A. \end{aligned}$$

Hence

$$B = -L$$
$$A = 0.$$

Then, from the time of release until the time when the marble leaves the piston, the motion is described by the equations

$$x(t) = -L \cos \omega t \qquad 3$$
$$v(t) = \omega L \sin \omega t. \qquad 4$$

When do the marble and piston lose contact? The piston can only push, not pull, on the marble, and when the piston begins to slow down, contact is lost and the marble moves on at a constant velocity. From Eq. (4), we see that the time t_m at which the velocity reaches a maximum is given by

$$\omega t_m = \frac{\pi}{2}.$$

Substituting this in Eq. (3), we find

$$\begin{aligned} x(t_m) &= -L \cos \frac{\pi}{2} \\ &= 0. \end{aligned}$$

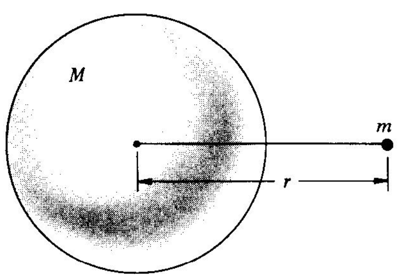
The marble loses contact as the spring passes its equilibrium point, as we expect, since the spring force retards the piston for $x > 0$.

From Eq. (4), the final speed of the marble is

$$v_{\max} = v(t_m)$$
$$= \omega L \sin\frac{\pi}{2}$$
$$= \sqrt{\frac{k}{m+M}}\,L.$$

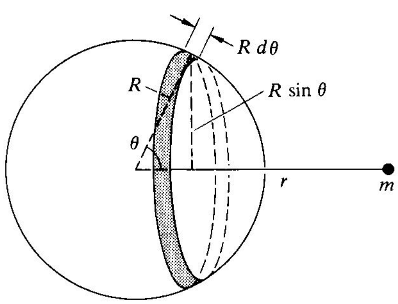
For the highest speeds, k and L should be large and $m + M$ should be small.

Note 2.1 The Gravitational Attraction of a Spherical Shell



In this note we calculate the gravitational force between a uniform thin spherical shell of mass M and a particle of mass m located a distance r from its center. We shall show that the magnitude of the force is GMm/r^2 if the particle is outside the shell and zero if the particle is inside.

To attack the problem, we divide the shell into narrow rings and add their forces by using integral calculus. Let R be the radius of the shell and t its thickness, $t \ll R$. The ring at angle θ, which subtends angle $d\theta$, has circumference $2\pi R \sin\theta$, width $R\,d\theta$, and thickness t. Its volume is



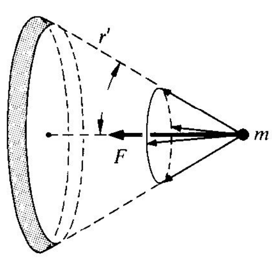
$$dV = 2\pi R^2 t \sin\theta\,d\theta$$

and its mass is

$$\rho\,dV = 2\pi R^2 t\rho \sin\theta\,d\theta$$
$$= \frac{M}{2}\sin\theta\,d\theta,$$

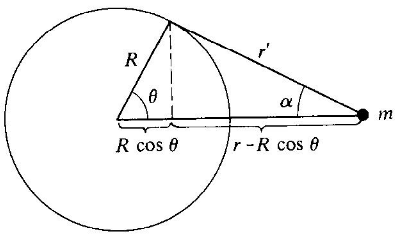
where $\rho = M/(4\pi R^2 t)$ is the density of the shell.

Each part of the ring is the same distance r' from m. The force on m due to a small section of the ring points toward that section. By symmetry, the transverse force components for the whole ring add vectorially to zero. Since the angle α between the force vector and the line of centers is the same for all sections of the ring, the force components along the line of centers add to give



$$dF = \frac{Gm\rho\,dV}{r'^2}\cos\alpha$$

for the whole ring.



The force due to the entire shell is

$$F = \int dF$$
$$= \int \frac{Gm\rho\, dV}{r'^2} \cos\alpha.$$

The problem now is to express all the quantities in the integrand in terms of one variable, say the polar angle θ. From the sketch, $\cos\alpha = (r - R\cos\theta)/r'$, and $r' = \sqrt{r^2 + R^2 - 2rR\cos\theta}$. Since

$$\rho\, dV = M \sin\theta\, d\theta/2,$$

we have

$$F = \left(\frac{GMm}{2}\right) \int_0^\pi \frac{(r - R\cos\theta)\sin\theta\, d\theta}{(r^2 + R_2 - 2rR\cos\theta)^{\frac{3}{2}}}.$$

A convenient substitution for evaluating this integral is $u = r - R\cos\theta$, $du = R\sin\theta\, d\theta$. Hence

$$F = \left(\frac{GMm}{2R}\right) \int_{r-R}^{r+R} \frac{u\, du}{(R^2 - r^2 + 2ru)^{\frac{3}{2}}}. \qquad 1$$

This integral is listed in standard tables. The result is

$$F = \frac{GMm}{2R}\frac{1}{2r^2}\left(\sqrt{R^2 - r^2 + 2ru} - \frac{r^2 - R^2}{\sqrt{R^2 - r^2 + 2ru}}\right)\Bigg|_{r-R}^{r+R}$$
$$= \frac{GMm}{4Rr^2}\left[(r + R) - (r - R) - (r^2 - R^2)\left(\frac{1}{r + R} - \frac{1}{r - R}\right)\right]$$
$$= \frac{GMm}{r^2} \qquad r > R.$$

For $r > R$, the shell acts gravitationally as though all its mass were concentrated at its center.

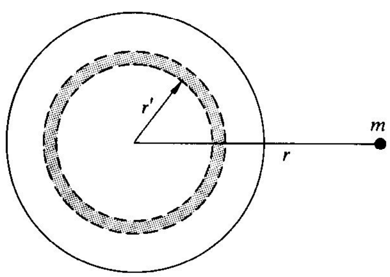
There is one subtlety in our evaluation of the integral. The term $\sqrt{r^2 + R^2 - 2rR}$ is inherently positive, and we must take

$$\sqrt{r^2 + R^2 - 2rR} = r - R,$$

since $r > R$. If the particle is inside the shell, the magnitude of the force is still given by Eq. (1). However, in this case $r < R$, and we must take $\sqrt{r^2 + R^2 - 2rR} = R - r$ in the evaluation. We find

$$F = \frac{GMm}{4Rr^2}\left[(R + r) - (R - r) - (r^2 - R^2)\left(\frac{1}{R + r} - \frac{1}{R - r}\right)\right]$$
$$= 0 \qquad r < R.$$

A solid sphere can be thought of as a succession of spherical shells. It is not hard to extend our results to this case when the density of the sphere $\rho(r')$ is a function only of radial distance r' from the center of



the sphere. The mass of a spherical shell of radius r' and thickness dr' is $\rho(r')4\pi r'^2\,dr'$. The force it exerts on m is

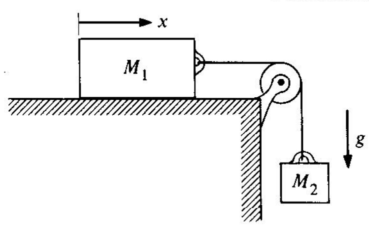
$$dF = \frac{Gm}{r^2}\rho(r')4\pi r'^2\,dr'.$$

Since the force exerted by every shell is directed toward the center of the sphere, the total force is

$$F = \frac{Gm}{r^2}\int_0^R \rho(r')4\pi r'^2\,dr'.$$

However, the integral is simply the total mass of the sphere, and we find that for $r > R$, the force between m and the sphere is identical to the force between two particles separated a distance r.

Problems

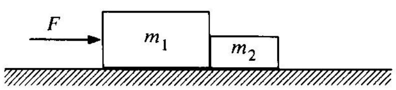


2.1 A 5-kg mass moves under the influence of a force $\mathbf{F} = (4t^2\hat{\mathbf{i}} - 3t\hat{\mathbf{j}})$ N, where t is the time in seconds (1 N = 1 newton). It starts from the origin at $t = 0$. Find: (*a*) its velocity; (*b*) its position; and (*c*) $\mathbf{r} \times \mathbf{v}$, for any later time.

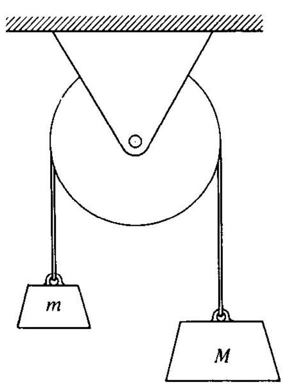
Ans. clue. (c) If $t = 1$ s, $\mathbf{r} \times \mathbf{v} = 6.7 \times 10^{-3}\hat{\mathbf{k}}$ m²/s

2.2 The two blocks shown in the sketch are connected by a string of negligible mass. If the system is released from rest, find how far block M_1 slides in time t. Neglect friction.

Ans. clue. If $M_1 = M_2$, $x = gt^2/4$



2.3 Two blocks are in contact on a horizontal table. A horizontal force is applied to one of the blocks, as shown in the drawing. If $m_1 = 2$ kg, $m_2 = 1$ kg, and $F = 3$ N, find the force of contact between the two blocks.



2.4 Two particles of mass m and M undergo uniform circular motion about each other at a separation R under the influence of an attractive force F. The angular velocity is ω radians per second. Show that $R = (F/\omega^2)(1/m + 1/M)$.

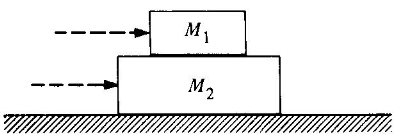
2.5 The Atwood's machine shown in the drawing has a pulley of negligible mass. Find the tension in the rope and the acceleration of M.

Ans. clue. If $M = 2m$, $T = \frac{4}{3}Mg$, $A = \frac{1}{3}g$

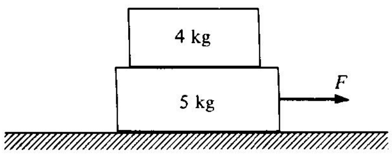
2.6 In a concrete mixer, cement, gravel, and water are mixed by tumbling action in a slowly rotating drum. If the drum spins too fast the ingredients stick to the drum wall instead of mixing.

Assume that the drum of a mixer has radius R and that it is mounted with its axle horizontal. What is the fastest the drum can rotate without the ingredients sticking to the wall all the time? Assume $g = 32$ ft/s².

Ans. clue. If $R = 2$ ft, $\omega_{max} = 4$ rad/s ≈ 38 rotations per minute

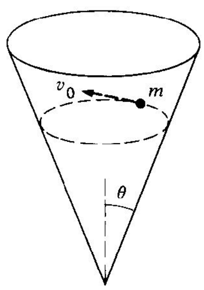


2.7 A block of mass M_1 rests on a block of mass M_2 which lies on a frictionless table. The coefficient of friction between the blocks is μ. What is the maximum horizontal force which can be applied to the blocks for them to accelerate without slipping on one another if the force is applied to (*a*) block 1 and (*b*) block 2?



2.8 A 4-kg block rests on top of a 5-kg block, which rests on a frictionless table. The coefficient of friction between the two blocks is such that the blocks start to slip when the horizontal force F applied to the lower block is 27 N. Suppose that a horizontal force is now applied only to the upper block. What is its maximum value for the blocks to slide without slipping relative to each other?

Ans. $F = 21.6$ N



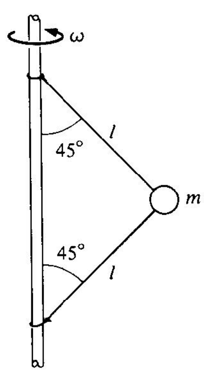
2.9 A particle of mass m slides without friction on the inside of a cone. The axis of the cone is vertical, and gravity is directed downward. The apex half-angle of the cone is θ, as shown.

The path of the particle happens to be a circle in a horizontal plane. The speed of the particle is v_0.

Draw a force diagram and find the radius of the circular path in terms of v_0, g, and θ.

2.10 Find the radius of the orbit of a synchronous satellite which circles the earth. (A synchronous satellite goes around the earth once every 24 h, so that its position appears stationary with respect to a ground station.) The simplest way to find the answer and give your results is by expressing all distances in terms of the earth's radius.

Ans. $6.6R_e$

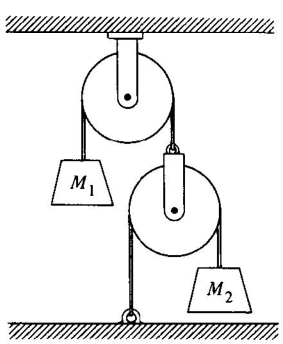


2.11 A mass m is connected to a vertical revolving axle by two strings of length l, each making an angle of 45° with the axle, as shown. Both the axle and mass are revolving with angular velocity ω. Gravity is directed downward.

a. Draw a clear force diagram for m.

b. Find the tension in the upper string, T_{up}, and lower string, T_{low}.

Ans. clue. If $l\omega^2 = \sqrt{2}\,g$, $T_{up} = \sqrt{2}\,mg$

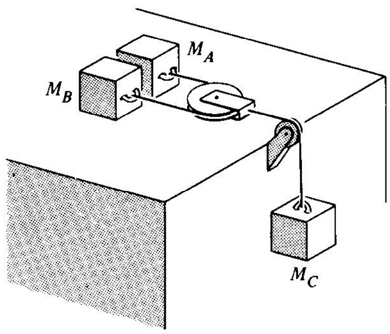


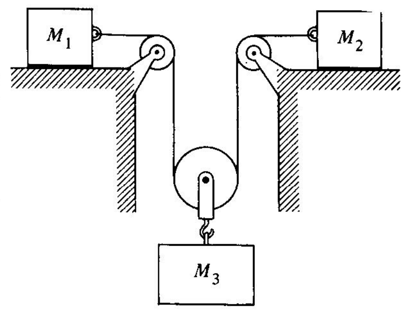
2.12 If you have courage and a tight grip, you can yank a tablecloth out from under the dishes on a table. What is the longest time in which the cloth can be pulled out so that a glass 6 in from the edge comes to rest before falling off the table? Assume that the coefficient of friction of the glass sliding on the tablecloth or sliding on the tabletop is 0.5. (For the trick to be effective the cloth should be pulled out so rapidly that the glass does not move appreciably.)

2.13 Masses M_1 and M_2 are connected to a system of strings and pulleys as shown. The strings are massless and inextensible, and the pulleys are massless and frictionless. Find the acceleration of M_1.

Ans. clue. If $M_1 = M_2$, $\ddot{x}_1 = g/5$

2.14 Two masses, A and B, lie on a frictionless table (see below left). They are attached to either end of a light rope of length l which passes around a pulley of negligible mass. The pulley is attached to a rope connected to a hanging mass, C. Find the acceleration of each mass. (You can check whether or not your answer is reasonable by considering special cases—for instance, the cases $M_A = 0$, or $M_A = M_B = M_C$.)





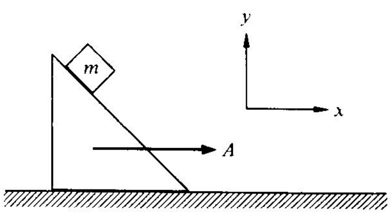
2.15 The system on the right above uses massless pulleys and rope. The coefficient of friction between the masses and horizontal surfaces is μ. Assume that M_1 and M_2 are sliding. Gravity is directed downward

a. Draw force diagrams, and show all relevant coordinates.

b. How are the accelerations related?

c. Find the tension in the rope, T.

Ans. $T = (\mu + 1)g/[2/M_3 + 1/(2M_1) + 1/(2M_2)]$



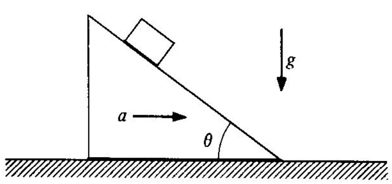
2.16 A 45° wedge is pushed along a table with constant acceleration A. A block of mass m slides without friction on the wedge. Find its acceleration. (Gravity is directed down.)

Ans. clue. If $A = 3g$, $\ddot{y} = g$

2.17 A block rests on a wedge inclined at angle θ. The coefficient of friction between the block and plane is μ.

a. Find the maximum value of θ for the block to remain motionless on the wedge when the wedge is fixed in position.

Ans. $\tan\theta = \mu$



b. The wedge is given horizontal acceleration a, as shown. Assuming that $\tan\theta < \mu$, find the minimum acceleration for the block to remain on the wedge without sliding.

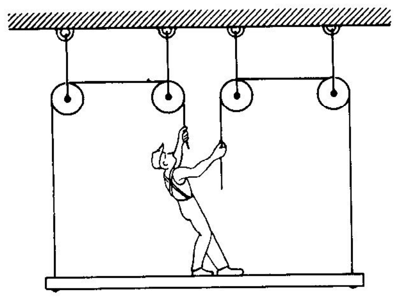
Ans. clue. If $\theta = \pi/4$, $a_{\min} = g(1 - \mu)/(1 + \mu)$

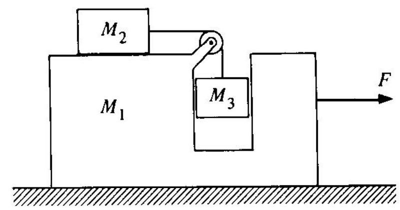
c. Repeat part *b*, but find the maximum value of the acceleration.

Ans. clue. If $\theta = \pi/4$, $a_{\max} = g(1 + \mu)/(1 - \mu)$

2.18 A painter of mass M stands on a platform of mass m and pulls himself up by two ropes which hang over pulleys, as shown. He pulls each rope with force F and accelerates upward with a uniform acceleration a. Find a—neglecting the fact that no one could do this for long.

Ans. clue. If $M = m$ and $F = Mg$, $a = g$



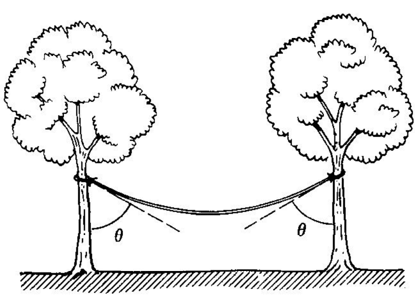


2.19 A "Pedagogical Machine" is illustrated in the sketch above. All surfaces are frictionless. What force F must be applied to M_1 to keep M_3 from rising or falling?

Ans. clue. For equal masses, $F = 3Mg$

2.20 Consider the "Pedagogical Machine" of the last problem in the case where F is zero. Find the acceleration of M_1.

Ans. $a_1 = -M_2M_3g/(M_1M_2 + M_1M_3 + 2M_2M_3 + M_3^2)$



2.21 A uniform rope of mass m and length l is attached to a block of mass M. The rope is pulled with force F. Find the tension at distance x from the end of the rope. Neglect gravity.

2.22 A uniform rope of weight W hangs between two trees. The ends of the rope are the same height, and they each make angle θ with the trees. Find

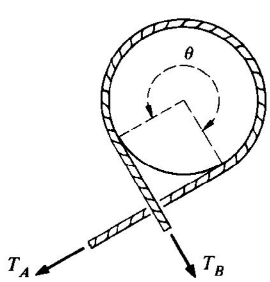
a. The tension at either end of the rope

b. The tension in the middle of the rope

Ans. clue. If $\theta = 45°$, $T_{\text{end}} = W/\sqrt{2}$, $T_{\text{middle}} = W/2$

2.23 A piece of string of length l and mass M is fastened into a circular loop and set spinning about the center of a circle with uniform angular velocity ω. Find the tension in the string. Suggestion: Draw a force diagram for a small piece of the loop subtending a small angle, $\Delta\theta$.

Ans. $T = M\omega^2 l/(2\pi)^2$



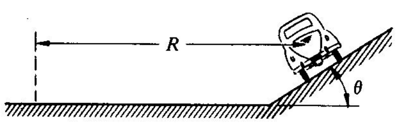
2.24 A device called a capstan is used aboard ships in order to control a rope which is under great tension. The rope is wrapped around a fixed drum, usually for several turns (the drawing shows about three-fourths turn). The load on the rope pulls it with a force T_A, and the sailor holds it with a much smaller force T_B. Can you show that $T_B = T_A e^{-\mu\theta}$, where μ is the coefficient of friction and θ is the total angle subtended by the rope on the drum?

2.25 Find the shortest possible period of revolution of two identical gravitating solid spheres which are in circular orbit in free space about a point midway between them. (You can imagine the spheres fabricated from any material obtainable by man.)

2.26 The gravitational force on a body located at distance R from the center of a uniform spherical mass is due solely to the mass lying at distance $r \leq R$, measured from the center of the sphere. This mass exerts a force as if it were a point mass at the origin.

Use the above result to show that if you drill a hole through the earth and then fall in, you will execute simple harmonic motion about the earth's center. Find the time it takes you to return to your point of departure and show that this is the time needed for a satellite to circle the earth in a low orbit with $r \approx R_e$. In deriving this result, you need to treat the earth as a uniformly dense sphere, and you must neglect all friction and any effects due to the earth's rotation.

2.27 As a variation of the last problem, show that you will also execute simple harmonic motion with the same period even if the straight hole passes far from the earth's center.



2.28 An automobile enters a turn whose radius is R. The road is banked at angle θ, and the coefficient of friction between wheels and road is μ. Find the maximum and minimum speeds for the car to stay on the road without skidding sideways.

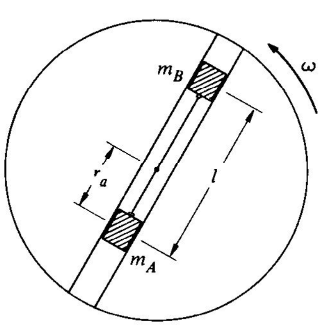
Ans. clue. If $\mu = 1$ and $\theta = \pi/4$, all speeds are possible

2.29 A car is driven on a large revolving platform which rotates with constant angular speed ω. At $t = 0$ a driver leaves the origin and follows a line painted radially outward on the platform with constant speed v_0. The total weight of the car is W, and the coefficient of friction between the car and stage is μ.

a. Find the acceleration of the car as a function of time using polar coordinates. Draw a clear vector diagram showing the components of *acceleration* at some time $t > 0$.

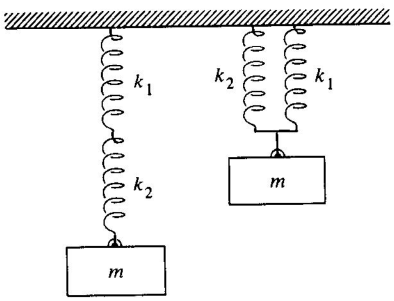
b. Find the time at which the car just starts to skid.

c. Find the direction of the friction force with respect to the instantaneous position vector **r** just before the car starts to skid. Show your result on a clear diagram.



2.30 A disk rotates with constant angular velocity ω, as shown. Two masses, m_A and m_B, slide without friction in a groove passing through the center of the disk. They are connected by a light string of length l, and are initially held in position by a catch, with mass m_A at distance r_A from the center. Neglect gravity. At $t = 0$ the catch is removed and the masses are free to slide.

Find $\ddot{r}_A$ immediately after the catch is removed in terms of m_A, m_B, l, r_A, and ω.



(a) (b)

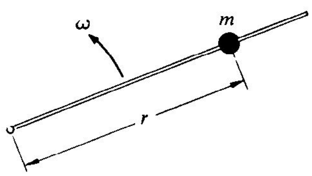
2.31 Find the frequency of oscillation of mass m suspended by two springs having constants k_1 and k_2, in each of the configurations shown.

Ans. clue. If $k_1 = k_2 = k$, $\omega_a = \sqrt{k/2m}$, $\omega_b = \sqrt{2k/m}$

2.32 A wheel of radius R rolls along the ground with velocity V. A pebble is carefully released on top of the wheel so that it is instantaneously at rest on the wheel.

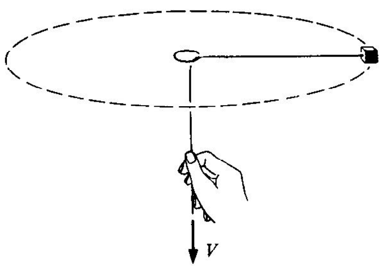
a. Show that the pebble will immediately fly off the wheel if $V > \sqrt{Rg}$.

b. Show that in the case where $V < \sqrt{Rg}$, and the coefficient of friction is $\mu = 1$, the pebble starts to slide when it has rotated through an angle given by $\theta = \arccos\,[(1/\sqrt{2})(V^2/Rg)] - \pi/4$.



2.33 A particle of mass m is free to slide on a thin rod. The rod rotates in a plane about one end at constant angular velocity ω. Show that the motion is given by $r = Ae^{-\gamma t} + Be^{+\gamma t}$, where γ is a constant which you must find and A and B are arbitrary constants. Neglect gravity.

Show that for a particular choice of initial conditions [that is, $r(t = 0)$ and $v(t = 0)$], it is possible to obtain a solution such that r decreases continually in time, but that for any other choice r will eventually increase. (Exclude cases where the bead hits the origin.)

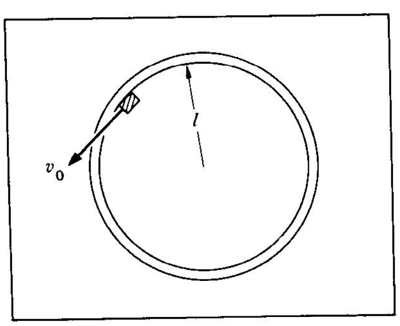


2.34. A mass m whirls around on a string which passes through a ring, as shown. Neglect gravity. Initially the mass is distance r_0 from the center and is revolving at angular velocity ω_0. The string is pulled with constant velocity V starting at $t = 0$ so that the radial distance to the mass decreases. Draw a force diagram and obtain a differential equation for ω. This equation is quite simple and can be solved either by inspection or by formal integration. Find

a. $\omega(t)$.

Ans. clue. For $Vt = r_0/2$, $\omega = 4\omega_0$

b. The force needed to pull the string.



2.35 This problem involves solving a simple differential equation.

A block of mass m slides on a frictionless table. It is constrained to move inside a ring of radius l which is fixed to the table. At $t = 0$, the block is moving along the inside of the ring (i.e., in the tangential direction) with velocity v_0. The coefficient of friction between the block and the ring is μ.

a. Find the velocity of the block at later times.

Ans. $v_0/[1 + (\mu v_0 t/l)]$

b. Find the position of the block at later times.

2.36 This problem involves a simple differential equation. You should be able to integrate it after a little "playing around."

A particle of mass m moving along a straight line is acted on by a retarding force (one always directed against the motion) $F = be^{\alpha v}$, where

b and α are constants and v is the velocity. At $t = 0$ it is moving with velocity v_0. Find the velocity at later times.

Ans. $v(t) = (1/\alpha) \ln [1/(\alpha b t/m + e^{-\alpha v_0})]$

2.37 The Eureka Hovercraft Corporation wanted to hold hovercraft races as an advertising stunt. The hovercraft supports itself by blowing air downward, and has a big fixed propeller on the top deck for forward propulsion. Unfortunately, it has no steering equipment, so that the pilots found that making high speed turns was very difficult. The company decided to overcome this problem by designing a bowl shaped track in which the hovercraft, once up to speed, would coast along in a circular path with no need to steer. They hired an engineer to design and build the track, and when he finished, he hastily left the country. When the company held their first race, they found to their dismay that the craft took exactly the same time T to circle the track, no matter what its speed. Find the equation for the cross section of the bowl in terms of T.