

# Interacting stochastic processes

Stefan Grosskinsky

Warwick, 2009

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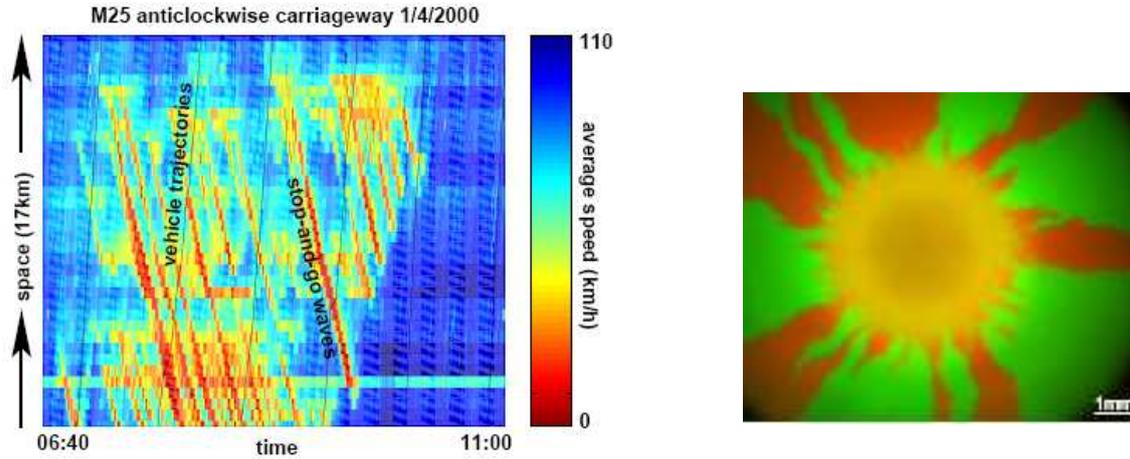
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## Introduction

Interacting particle systems (IPS) are mathematical models of complex phenomena involving a large number of interrelated components. There are numerous examples within all areas of natural and social sciences, such as traffic flow on motorways or communication networks, opinion dynamics, spread of epidemics or fires, genetic evolution, reaction diffusion systems, crystal surface growth, financial markets, etc. The central question is to understand and predict emergent behaviour on macroscopic scales, as a result of the microscoping dynamics and interactions of individual components. Qualitative changes in this behaviour depending on the system parameters are known as collective phenomena or phase transitions and are of particular interest.

In IPS the components are modeled as particles confined to a lattice or some discrete geometry. But applications are not limited to systems endowed with such a geometry, since continuous degrees of freedom can often be discretized without changing the main features. So depending on the specific case, the particles can represent cars on a motorway, molecules in ionic channels, or prices of asset orders in financial markets, to name just a few examples. In principle such systems often evolve according to well-known laws, but in many cases microscopic details of motion are not fully accessible. Due to the large system size these influences on the dynamics can be approximated as effective random noise with a certain postulated distribution. The actual origin of the noise, which may be related to chaotic motion or thermal interactions, is usually ignored. On this level the statistical description in terms of a random process where particles move and interact according to local stochastic rules is an appropriate mathematical model. It is neither possible nor required to keep track of every single particle. One is rather interested in predicting measurable quantities which correspond to expected values of certain observables, such as the growth rate of the crystalline surface or the flux of cars on the motorway. Although describing the system only on a mesoscopic level as explained above, stochastic particle systems are usually referred to as microscopic models and we stick to this convention. On a macroscopic scale, a continuum description of systems with a large number of particles is given by coarse-grained density fields, evolving in time according to a partial differential equation. The form of this equation depends on the particular application, and its mathematical connection to microscopic particle models is one of the fundamental questions in complexity science.

The focus of these notes is not on detailed models of real world phenomena, but on simple paradigmatic IPS that capture the main features of real complex systems. Several such models have been introduced in the seminal paper [1]. They allow for a detailed mathematical analysis leading to a deeper understanding of the fundamental principles of emergence and collective phenomena. The notes provide an introduction into the well developed mathematical theory for the description of the dynamics of IPS, involving graphical representations and an analytic description using semigroups and generators. Since the external conditions for real systems are often constant or slowly varying with respect to the system dynamics, observations are typically available in a time stationary situation. This is described by the invariant measures of the stochastic particle systems which are thus of major interest in their analysis, and are introduced at the end of Section 1 which covers the basic theory. Later sections provide a more detailed discussion of several examples of basic IPS, probabilistic techniques used for their analysis such as coupling, duality and relative entropy methods are introduced along. The second main aspect of the notes, also covered in Sections 2 to 4, is to get acquainted with different types of collective phenomena in complex systems. We will discuss their intimate relationship with symmetries and conservation laws of the dynamics, and make connections to the classical theory of phase transitions in statistical mechanics.



**Figure 1:** Left: Traffic data gathered from inductance loops showing car speeds on the M25 London orbital motorway averaged over 1 minute intervals (taken from [2]). Right: segregation pattern of a neutral genetic marker in *E. coli* growth from a mixed population of circular shape (taken from [3]).

Necessary prerequisites for the reader are basic knowledge in undergraduate mathematics, in particular probability theory and stochastic processes. For the latter, discrete time Markov chains are sufficient since the concept of continuous time processes will be introduced in Section 1. Acquaintance with measure theoretic concepts and basic functional analysis is helpful but not necessary.

Before we get immersed into mathematics let us quickly discuss two real world examples of recent studies, to illustrate the motivation and the origin of the processes which we will work with. The left of Figure 1 shows colour-coded data for the car speed on the M25 London orbital motorway as a function of space and time. The striped patterns of low speed correspond to stop-and-go waves during rush hour. Often there is no obvious external cause such as an accident or road closure, so this pattern has to be interpreted as an intrinsic collective phenomenon emerging from the interactions of cars on a busy road. A minimal mathematical description of this situation in terms of IPS would be to take a one-dimensional lattice  $\Lambda = \mathbb{Z}$  (or a subset thereof), and at each site  $x \in \Lambda$  denote the presence or absence of a car with an occupation number  $\eta(x) = 1$  or 0, respectively. So the state space of our mathematical model is given by the set  $\{0, 1\}^\Lambda$  denoting all possible configurations  $\eta = (\eta(x))_{x \in \Lambda}$ . In terms of dynamics, we only want to model normal traffic on a single lane road without car crashes or overtaking. So cars are allowed to proceed one lattice site to the right, say, with a given rate<sup>1</sup>, provided that the site in front is not occupied by another car. The rate may depend on the surrounding configuration of cars (e.g. number of empty sites ahead), and relatively simple choices depending only on three or four neighbouring sites can already lead to interesting patterns and the emergence of stop-and-go waves. There are numerous such approaches in the literature known as cellular automata models for traffic flow, see e.g. [4] and references therein. The defining features of this process in terms of IPS are that no particles are created or destroyed (conservation of the number of particles) and that there is at most one particle per site (exclusion rule). Processes with both or only the first property will be discussed in detail in Sections 2 and 3, respectively.

The right of Figure 1 shows segregation patterns of a microbial species (*E. coli*) when grown in

<sup>1</sup>The concept of 'rate' and exact mathematical formulations of the dynamics will be introduced in Section 1.

rich growth medium from a mixed initial population of circular shape. Each individual appears either red or green as a result of a neutral genetic marker that only affects the colour. A possible IPS model of this system has a state space  $\{0, R, G\}^\Lambda$ , where the lattice is now two dimensional, say  $\Lambda = \mathbb{Z}^2$  for simplicity, and 0 represents an empty site,  $R$  the presence of a red and  $G$  of a green individual. The dynamics can be modeled by letting each individual split into two ( $R \rightarrow 2R$  or  $G \rightarrow 2G$ ) with a given rate, and then place the offspring on an empty neighbouring site. If there is no empty neighbouring site the reproduction rate is zero (or equivalently the offspring is immediately killed). Therefore we have two equivalent species competing for the same resource (empty sites), and spatial segregation is a result of the fact that once the red particles died out in a certain region due to fluctuations, all the offspring is descending from green ancestors. Note that in contrast to the first example, the number of particles in this model is not conserved. The simplest such process to model extinction or survival of a single species is called contact process, and is discussed in detail in Section 4.

# 1 Basic theory

## 1.1 Markov processes

The *state space*  $X$  of a stochastic process is the set of all *configurations* which we typically denote by  $\eta$  or  $\zeta$ . For interacting particle systems the state space is of the form  $X = S^\Lambda$  where the *local state space*  $S \subseteq \mathbb{Z}$  is a finite subset of the integers, such as  $S = \{0, 1\}$  or  $S = \{0, 1, 2\}$  to indicate e.g. the local occupation number  $\eta(x) \in S$  for all  $x \in \Lambda$ .  $\Lambda$  is any countable set such as a regular lattice or the vertex set of a graph. We often do not explicitly specify the edge set or connectivity structure of  $\Lambda$  but, unless stated otherwise, we always assume it to be strongly connected to avoid degeneracies, i.e. any pair of points in  $\Lambda$  is connected (along a possibly directed path).

The particular structure of the state space is not essential to define a Markov process in general, what is essential for that is that  $X$  is a compact metric space. Spaces of the above form have that property w.r.t. the product topology<sup>1</sup>. The metric structure of  $X$  allows us to properly define regular sample paths and continuous functions, and set up a measurable structure, whereas compactness becomes important only later in connection with stationary distributions.

A *continuous time stochastic process*  $(\eta_t : t \geq 0)$  is then a family of random variables  $\eta_t$  taking values in  $X$ . This can be characterized by a probability measure  $\mathbb{P}$  on the *canonical path space*

$$D[0, \infty) = \{\eta : [0, \infty) \rightarrow X \text{ càdlàg}\} . \quad (1.3)$$

By convention, this is the set of right continuous functions with left limits (càdlàg). The elements of  $D[0, \infty)$  are the *sample paths*  $t \mapsto \eta_t \in X$ , written shortly as  $\eta$ . or  $\boldsymbol{\eta}$ . For an IPS with  $X = S^\Lambda$ ,  $\eta_t(x)$  denotes the occupation of site  $x$  at time  $t$ .

Note that as soon as  $|S| > 1$  and  $\Lambda$  is infinite the state space  $X = S^\Lambda$  is uncountable. But even if  $X$  itself is countable (e.g. for finite lattices), the path space is always uncountable due to the continuous time  $t \in \mathbb{R}$ . Therefore we have to think about measurable structures on  $D[0, \infty)$  and the state space  $X$  in the following. The technical details of this are not really essential for the understanding but we include them here for completeness. The metric on  $X$  provides us with a generic topology of open sets generating the Borel  $\sigma$ -algebra, which we take as the measurable structure on  $X$ . Now, let  $\mathcal{F}$  be the smallest  $\sigma$ -algebra on  $D[0, \infty)$  such that all the mappings  $\eta \mapsto \eta_s$  for  $s \geq 0$  are measurable w.r.t.  $\mathcal{F}$ . That means that every path can be evaluated or observed at arbitrary times  $s$ , i.e.

$$\{\eta_s \in A\} = \{\eta \mid \eta_s \in A\} \in \mathcal{F} \quad (1.4)$$

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<sup>1</sup>Why is  $X = S^\Lambda$  a compact metric space?

The discrete topology  $\sigma_x$  on the local state space  $S$  is simply given by the power set, i.e. all subsets are 'open'. The choice of the metric does not influence this and is therefore irrelevant for that question. The *product topology*  $\sigma$  on  $X$  is then given by the smallest topology such that all the canonical projections  $\eta(x) : X \rightarrow S$  (occupation at a site  $x$  for a given configuration  $\eta$ ) are continuous (pre-images of open sets are open). That means that  $\sigma$  is generated by sets

$$\eta(x)^{-1}(U) = \{\eta : \eta(x) \in U\} , \quad U \subseteq \{0, 1\} , \quad (1.1)$$

which are called *open cylinders*. Finite intersections of these sets

$$\{\eta : \eta(x_1) \in U_1, \dots, \eta(x_n) \in U_n\} , \quad n \in \mathbb{N}, U_i \subseteq \{0, 1\} \quad (1.2)$$

are called *cylinder sets* and any open set on  $X$  is a (finite or infinite) union of cylinder sets. Clearly  $\{0, 1\}$  is compact since  $\sigma_x$  is finite, and by *Tychonoff's theorem* any product of compact topological spaces is compact (w.r.t. the product topology). This holds for any countable lattice or vertex set  $\Lambda$ .

for all measurable subsets  $A \in X$ . This is certainly a reasonable minimal requirement for  $\mathcal{F}$ . If  $\mathcal{F}_t$  is the smallest  $\sigma$ -algebra on  $D[0, \infty)$  relative to which all the mappings  $\eta \mapsto \eta_s$  for  $s \leq t$  are measurable, then  $(\mathcal{F}_t : t \geq 0)$  provides a natural filtration for the process. The filtered space  $(D[0, \infty), \mathcal{F}, (\mathcal{F}_t : t \geq 0))$  serves as a generic choice for the probability space of a stochastic process.

**Definition 1.1** A (*homogeneous*) *Markov process* on  $X$  is a collection  $(\mathbb{P}^\zeta : \zeta \in X)$  of probability measures on  $D[0, \infty)$  with the following properties:

- (a)  $\mathbb{P}^\zeta(\eta \in D[0, \infty) : \eta_0 = \zeta) = 1$  for all  $\zeta \in X$ ,  
i.e.  $\mathbb{P}^\zeta$  is normalized on all paths with initial condition  $\eta_0 = \zeta$ .
- (b)  $\mathbb{P}^\zeta(\eta_{t+} \in A | \mathcal{F}_t) = \mathbb{P}^{\eta_t}(A)$  for all  $\zeta \in X$ ,  $A \in \mathcal{F}$  and  $t > 0$ . (Markov property)
- (c) The mapping  $\zeta \mapsto \mathbb{P}^\zeta(A)$  is measurable for every  $A \in \mathcal{F}$ .

Note that the Markov property as formulated in (b) implies that the process is (time-)homogeneous, since the law  $\mathbb{P}^{\eta_t}$  does not have an explicit time dependence. Markov processes can be generalized to be inhomogeneous (see e.g. [15]), but we will concentrate only on homogeneous processes. The condition in (c) allows to consider processes with general initial distributions  $\mu \in \mathcal{M}_1(X)$  via

$$\mathbb{P}^\mu := \int_X \mathbb{P}^\zeta \mu(d\zeta). \quad (1.5)$$

When we do not want to specify the initial condition for the process we will often only write  $\mathbb{P}$ .

## 1.2 Continuous time Markov chains and graphical representations

Throughout this section let  $X$  be a countable set. Markov processes on  $X$  are called *Markov chains*. They can be understood without a path space description on a more basic level, by studying the time evolution of distributions  $p_t(\eta) := \mathbb{P}(\eta_t = \zeta)$  (see e.g. [14] or [15]). The dynamics of Markov chains can be characterized by *transition rates*  $c(\zeta, \zeta') \geq 0$ , which have to be specified for all  $\zeta, \zeta' \in X$ . For a given process  $(\mathbb{P}^\zeta : \zeta \in X)$  the rates are defined via

$$\mathbb{P}^\zeta(\eta_t = \zeta') = c(\zeta, \zeta')t + o(t) \quad \text{as } t \searrow 0 \quad \text{for } \zeta \neq \zeta', \quad (1.6)$$

and represent probabilities per unit time. We do not go into the details here of why the linearization in (1.6) for small times  $t$  is valid. It can be shown under the assumption of uniform continuity of  $t \mapsto \mathbb{P}^\zeta(\eta_t = \zeta')$  as  $t \searrow 0$ , which is also called *strong continuity* (see e.g. [13], Section 19). This is discussed in more detail for general Markov processes in Section 1.4. We will see in the next subsection how a given set of rates determines the path measures of a process. Now we would like to get an intuitive understanding of the time evolution and the role of the transition rates. For a process with  $\eta_0 = \zeta$ , we denote by

$$W_\zeta := \inf\{t \geq 0 : \eta_t \neq \zeta\} \quad (1.7)$$

the *holding time* in state  $\zeta$ . The value of this time is related to the total exit rate out of state  $\zeta$ ,

$$c_\zeta := \sum_{\zeta' \neq \zeta} c(\zeta, \zeta'). \quad (1.8)$$

We assume in the following that  $c_\zeta < \infty$  for all  $\zeta \in X$  (which is only a restriction if  $X$  is infinite). As shown below, this ensures that the process has a well defined waiting time in each state  $c_\zeta$  which is essential to construct the dynamics locally in time. To have well defined global dynamics for all  $t \geq 0$  we also have to exclude that the chain explodes<sup>1</sup>, which is ensured by a uniform bound

$$\bar{c} = \sup_{\zeta \in X} c_\zeta < \infty. \quad (1.9)$$

If  $c_\zeta = 0$ ,  $\zeta$  is an *absorbing state* and  $W_\zeta = \infty$  a.s. .

**Proposition 1.1** *If  $c_\zeta \in (0, \infty)$ ,  $W_\zeta \sim \text{Exp}(c_\zeta)$  and  $\mathbb{P}^\zeta(\eta_{W_\zeta} = \zeta') = c(\zeta, \zeta')/c_\zeta$ .*

**Proof.**  $W_\zeta$  has the 'loss of memory' property

$$\mathbb{P}^\zeta(W_\zeta > s + t | W_\zeta > s) = \mathbb{P}^\zeta(W_\zeta > s + t | \eta_s = \zeta) = \mathbb{P}^\zeta(W_\zeta > t), \quad (1.10)$$

the distribution of the holding time  $W_\zeta$  does not depend on how much time the process has already spent in state  $\zeta$ . Thus

$$\mathbb{P}^\zeta(W_\zeta > s + t, W_\zeta > s) = \mathbb{P}^\zeta(W_\zeta > s + t) = \mathbb{P}^\zeta(W_\zeta > s) \mathbb{P}^\zeta(W_\zeta > t). \quad (1.11)$$

This is the functional equation for an exponential and implies that

$$\mathbb{P}^\zeta(W_\zeta > t) = e^{\lambda t} \quad (\text{with initial condition } \mathbb{P}^\zeta(W_\zeta > 0) = 1). \quad (1.12)$$

The exponent is given by

$$\lambda = \frac{d}{dt} \mathbb{P}^\zeta(W_\zeta > t) \Big|_{t=0} = \lim_{t \searrow 0} \frac{\mathbb{P}^\zeta(W_\zeta > t) - 1}{t} = -c_\zeta, \quad (1.13)$$

since with (1.6) and (1.8)

$$\mathbb{P}^\zeta(W_\zeta > 0) = 1 - \mathbb{P}^\zeta(\eta_t \neq \zeta) + o(t) = 1 - c_\zeta t + o(t). \quad (1.14)$$

Now, conditioned on a jump occuring in the time interval  $[t, t + h)$  we have

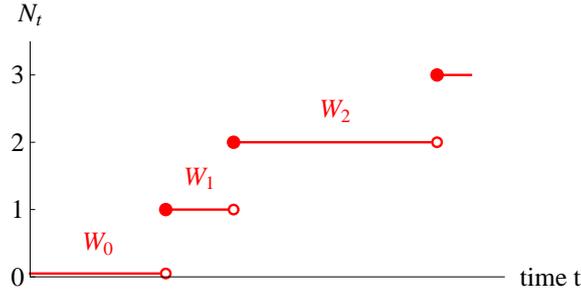
$$\mathbb{P}^\zeta(\eta_{t+h} = \zeta' | t \leq W_\zeta < t + h) = \mathbb{P}^\zeta(\eta_h = \zeta' | W_\zeta < h) = \frac{\mathbb{P}^\zeta(\eta_h = \zeta')}{\mathbb{P}^\zeta(W_\zeta < h)} \rightarrow \frac{c(\zeta, \zeta')}{c_\zeta} \quad (1.15)$$

as  $h \searrow 0$ , using the Markov property and L'Hopital's rule with (1.6) and (1.13). With right-continuity of paths, this implies the second statement.  $\square$

We summarize some important properties of exponential random variables, the proof of which can be found in any standard textbook. Let  $W_1, W_2, \dots$  be a sequence of independent exponentials  $W_i \sim \text{Exp}(\lambda_i)$ . Then  $\mathbb{E}(W_i) = 1/\lambda_i$  and

$$\min\{W_1, \dots, W_n\} \sim \text{Exp}\left(\sum_{i=1}^n \lambda_i\right). \quad (1.16)$$

<sup>1</sup>Explosion means that the Markov chain exhibits infinitely many jumps in finite time. For more details see e.g. [14], Section 2.7.



**Figure 2:** Sample path (càdlàg) of a Poisson process with holding times  $W_0, W_1, \dots$

The sum of iid exponentials with  $\lambda_i = \lambda$  is  $\Gamma$ -distributed, i.e.

$$\sum_{i=1}^n W_i \sim \Gamma(n, \lambda) \quad \text{with PDF} \quad \frac{\lambda^n w^{n-1}}{(n-1)!} e^{-\lambda w}. \quad (1.17)$$

**Example.** The *Poisson process*  $(N_t : t \geq 0)$  with rate  $\lambda > 0$  (short  $PP(\lambda)$ ) is a Markov chain with  $X = \mathbb{N} = \{0, 1, \dots\}$ ,  $N_0 = 0$  and  $c(n, m) = \lambda \delta_{n+1, m}$ .

With iidrv's  $W_i \sim Exp(\lambda)$  we can write  $N_t = \max\{n : \sum_{i=1}^n W_i \leq t\}$ . This implies

$$\begin{aligned} \mathbb{P}(N_t = n) &= \mathbb{P}\left(\sum_{i=1}^n W_i \leq t < \sum_{i=1}^{n+1} W_i\right) = \int_0^t \mathbb{P}\left(\sum_{i=1}^n W_i = s\right) \mathbb{P}(W_{n+1} > t-s) ds = \\ &= \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s} e^{-\lambda(t-s)} ds = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \end{aligned} \quad (1.18)$$

so  $N_t \sim Poi(\lambda t)$  has a Poisson distribution. Alternatively a Poisson process can be characterized by the following.

**Proposition 1.2**  $(N_t : t \geq 0) \sim PP(\lambda)$  if and only if it has stationary, independent increments, i.e.

$$N_{t+s} - N_s \sim N_t - N_0 \quad \text{and} \quad N_{t+s} - N_s \quad \text{independent of} \quad (N_u : u \leq s), \quad (1.19)$$

and for each  $t$ ,  $N_t \sim Poi(\lambda t)$ .

**Proof.** By the loss of memory property and (1.18) increments have the distribution

$$N_{t+s} - N_s \sim Poi(\lambda t) \quad \text{for all} \quad s \geq 0, \quad (1.20)$$

and are independent of  $N_s$  which is enough together with the Markov property.

The other direction follows by deriving the jump rates from the properties in (1.19) using (1.6).  $\square$

Remember that for independent Poisson variables  $Y_1, Y_2, \dots$  with  $Y_i \sim Poi(\lambda_i)$  we have  $\mathbb{E}(Y_i) = Var(Y_i) = \lambda_i$  and

$$\sum_{i=1}^n Y_i \sim Poi\left(\sum_{i=1}^n \lambda_i\right). \quad (1.21)$$

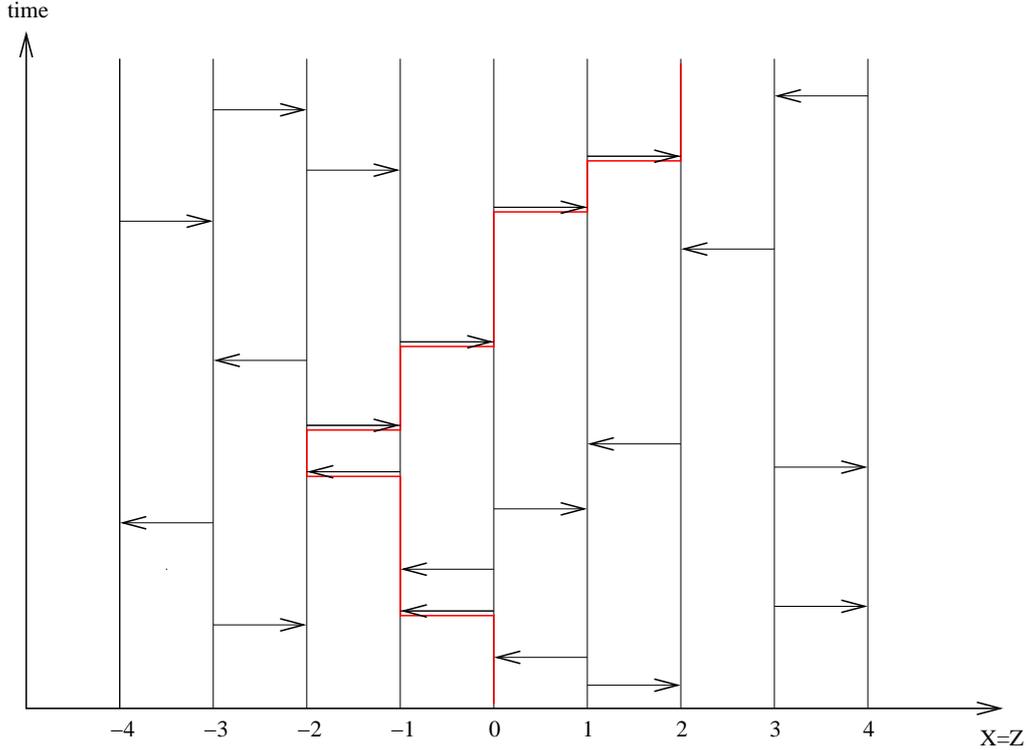
With Prop. 1.2 this immediately implies that adding a finite number of independent Poisson processes  $(N_t^i : t \geq 0) \sim PP(\lambda_i), i = 1, \dots, n$  results in a Poisson process, i.e.

$$M_t = \sum_{i=1}^n N_t^i \Rightarrow (M_t : t \geq 0) \sim PP\left(\sum_{i=1}^n \lambda_i\right). \quad (1.22)$$

**Example.** A continuous-time simple random walk  $(\eta_t : t \geq 0)$  on  $X = \mathbb{Z}$  with jump rates  $p$  to the right and  $q$  to the left is given by

$$\eta_t = R_t - L_t \quad \text{where} \quad (R_t : t \geq 0) \sim PP(p), (L_t : t \geq 0) \sim PP(q). \quad (1.23)$$

The process can be constructed by the following graphical representation:



In each column the arrows  $\rightarrow \sim PP(p)$  and  $\leftarrow \sim PP(q)$  are independent Poisson processes. Together with the initial condition, the trajectory of the process shown in red is then uniquely determined. An analogous construction is possible for a general Markov chain, which is a continuous time random walk on  $X$  with jump rates  $c(\zeta, \zeta')$ . In this way we can also construct interacting random walks and more general IPS, as is shown in the next section.

Note that the restriction  $c_\zeta > \infty$  for all  $\zeta \in X$  excludes e.g. random walks on  $X = \mathbb{Z}$  which move non-locally and jump to any site with rate  $c(\zeta, \zeta') = 1$ . In the graphical construction for such a process there would not be a well defined first jump event and the path could not be constructed. However, as long as the rates are summable, such as

$$c(\zeta, \zeta') = (\zeta - \zeta')^{-2} \quad \text{for all } \zeta, \zeta' \in \mathbb{Z}, \quad (1.24)$$

we have  $c_\zeta < \infty$ , and the basic properties of adding Poisson processes or taking minima of exponential random variables extend to infinitely many. So the process is well defined and the path can be constructed in the graphical representation.

### 1.3 Three basic IPS

For the IPS introduced in this section the state space is of the form  $X = \{0, 1\}^\Lambda$ , *particle configurations*  $\eta = (\eta(x) : x \in \Lambda)$ .  $\eta(x) = 1$  means that there is a particle at site  $x$  and if  $\eta(x) = 0$  site  $x$  is empty. The lattice  $\Lambda$  can be any countable set, typical examples we have in mind are regular lattices  $\Lambda = \mathbb{Z}^d$ , subsets of those, or the vertex set of a given graph.

As noted before, if  $\Lambda$  is infinite  $X$  is uncountable, so we are not dealing with Markov chains in this section. But for the processes we consider the particles move/interact only locally and one at a time, so a description with jump rates still makes sense. More specifically, for a given  $\eta \in X$  there are only countably many  $\eta'$  for which  $c(\eta, \eta') > 0$ . Define the configurations  $\eta^x$  and  $\eta^{xy} \in X$  for  $x \neq y \in \Lambda$  by

$$\eta^x(z) = \begin{cases} \eta(z) & , z \neq x \\ 1 - \eta(x) & , z = x \end{cases} \quad \text{and} \quad \eta^{xy}(z) = \begin{cases} \eta(z) & , z \neq x, y \\ \eta(y) & , z = x \\ \eta(x) & , z = y \end{cases}, \quad (1.25)$$

so that  $\eta^x$  corresponds to creation/annihilation of a particle at site  $x$  and  $\eta^{xy}$  to motion of a particle between  $x$  and  $y$ . Then following standard notation we write for the corresponding jump rates

$$c(x, \eta) = c(\eta, \eta^x) \quad \text{and} \quad c(x, y, \eta) = c(\eta, \eta^{xy}). \quad (1.26)$$

All other jump rates including e.g. multi-particle interactions or simultaneous motion are zero.

**Definition 1.2** Let  $p(x, y) \geq 0$ ,  $x, y \in \Lambda$ , be transition rates of an irreducible continuous-time random walk on  $\Lambda$ . The *exclusion process (EP)* on  $X$  is then characterized by the jump rates

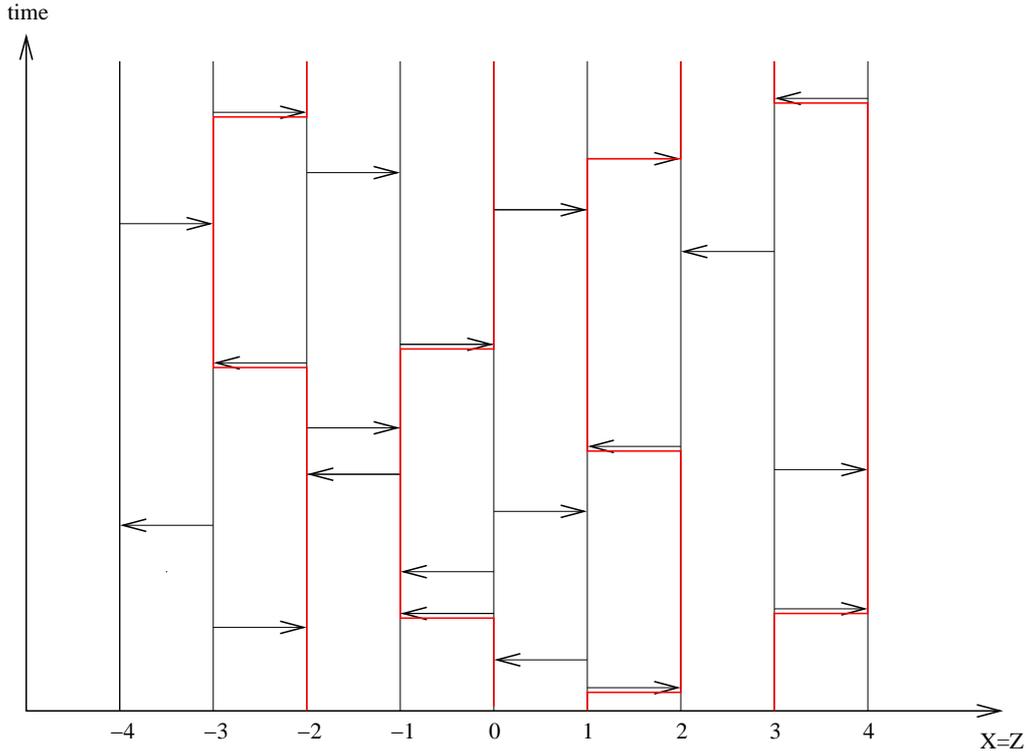
$$c(x, y, \eta) = p(x, y)\eta(x)(1 - \eta(y)), \quad x, y \in \Lambda \quad (1.27)$$

where particles only jump to empty sites (exclusion interaction). If  $\Lambda$  is a regular lattice and  $p(x, y) > 0$  only if  $x$  and  $y$  are nearest neighbours, the process is called *simple EP (SEP)*. If in addition  $p(x, y) = p(y, x)$  for all  $x, y \in \Lambda$  it is called *symmetric SEP (SSEP)* and otherwise *asymmetric SEP (ASEP)*.

Note that the presence of a direct connection (or directed edge)  $(x, y)$  is characterized by  $p(x, y) > 0$ , and irreducibility of  $p(x, y)$  is equivalent to  $\Lambda$  being strongly connected. Particles only move and are not created or annihilated, therefore the number of particles in the system is conserved in time. In general such IPS are called *lattice gases*. The ASEP in one dimension  $d = 1$  is one of the most basic and most studied models in IPS and nonequilibrium statistical mechanics (see e.g. [30] and references therein), and a common quick way of defining it is

$$10 \xrightarrow{p} 01, \quad 01 \xrightarrow{q} 10 \quad (1.28)$$

where particles jump to the right (left) with rate  $p$  ( $q$ ). Variants and extensions of exclusion processes are used to model all kinds of transport phenomena, including for instance traffic flow (see e.g. [30, 31] and references therein).



The graphical construction is analogous to the single particle process given above, with the additional constraint of the exclusion interaction. We will discuss exclusion processes in more detail in Section 2. Exclusion is of course not the only possible interaction between random walkers, and we will discuss a different example with a simpler zero-range interaction in Section 3.

**Definition 1.3** The *contact process (CP)* on  $X$  is characterized by the jump rates

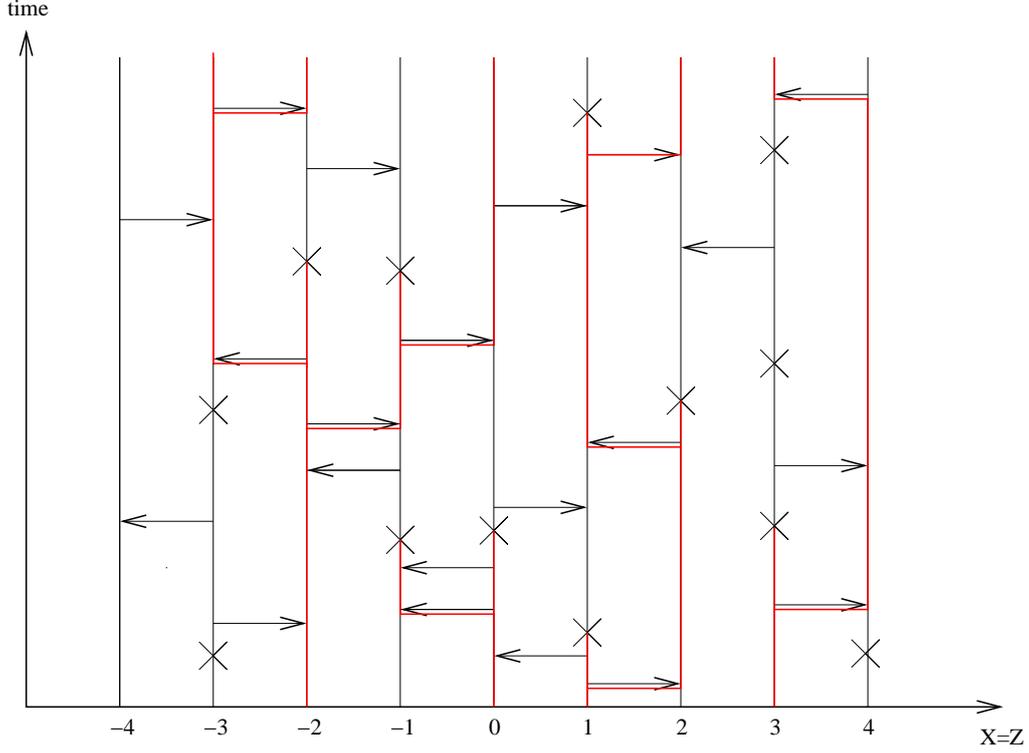
$$c(x, \eta) = \begin{cases} 1 & , \eta(x) = 1 \\ \lambda \sum_{y \sim x} \eta(y) & , \eta(x) = 0 \end{cases} , \quad x \in \Lambda . \quad (1.29)$$

Particles can be interpreted as infected sites which recover with rate 1 and are infected independently with rate  $\lambda > 0$  by particles on connected sites  $y \sim x$ .

In contrast to the EP the CP does not have a conserved quantity like the number of particles, but it does have an absorbing state  $\eta \equiv 0$ , since there is no spontaneous infection. A compact notation for the CP is

$$1 \xrightarrow{1} 0 , \quad 0 \rightarrow 1 \quad \text{with rate} \quad \lambda \sum_{y \sim x} \eta(y) . \quad (1.30)$$

The graphical construction below contains now a third independent Poisson process  $\times \sim PP(1)$  on each line marking the recovery events. The infection events are marked by the independent  $PP(\lambda)$  Poisson processes  $\rightarrow$  and  $\leftarrow$ .



The CP and related models have applications in population dynamics and the spread of infectious diseases/viruses etc. (see e.g. [32] and references therein).

**Definition 1.4** Let  $p(x, y) \geq 0$ ,  $x, y \in \Lambda$  be irreducible transition rates on  $\Lambda$  as for the EP. The *linear voter model (VM)* on  $X$  is characterized by the jump rates

$$c(x, \eta) = \sum_{y \in \Lambda} p(x, y) \left( \eta(x)(1 - \eta(y)) + (1 - \eta(x))\eta(y) \right), \quad x \in \Lambda. \quad (1.31)$$

0 and 1 can be interpreted as two different opinions, and a site  $x$  adopts the opinion of site  $y$  with rate  $p(x, y)$  independently for all connected sites with different opinion.

Note that the voter model is symmetric under flipping occupation numbers, i.e.

$$c(x, \eta) = c(x, \zeta) \quad \text{if} \quad \zeta(x) = 1 - \eta(x) \quad \text{for all } x \in \Lambda. \quad (1.32)$$

Consequently it has two absorbing states  $\eta \equiv 0, 1$ , which correspond to fixation of one of the opinions. For the general (non-linear) voter model the jump rates  $c(x, \eta)$  can be any function that exhibits the symmetry (1.32), no spontaneous change of opinion and monotonicity, i.e. for  $\eta(x) = 0$  we have

$$\begin{aligned} c(x, \eta) &= 0 & \text{if} & \quad \sum_{y \sim x} \eta(y) = 0, \\ c(x, \eta) &\geq c(x, \zeta) & \text{if} & \quad \eta(y) \geq \zeta(y) \quad \text{for all } y \sim x, \end{aligned} \quad (1.33)$$

with corresponding symmetric rules for  $\eta(x) = 1$ . This model and its generalizations have applications in opinion dynamics and formation of cultural beliefs (see e.g. [33] and references therein).

## 1.4 Semigroups and generators

Let  $X$  be a compact metric space and denote by

$$C(X) = \{f : X \rightarrow \mathbb{R} \text{ continuous}\} \quad (1.34)$$

the set of real-valued continuous functions. This as a Banach space with sup-norm  $\|f\|_\infty = \sup_{\eta \in X} |f(\eta)|$ , since by compactness of  $X$ ,  $\|f\|_\infty < \infty$  for all  $f \in C(X)$ . Functions  $f$  can be regarded as observables, and we are interested in their time evolution rather than the evolution of the full distribution. This is not only mathematically easier to formulate, but also more relevant in most applications. The full detail on the state of the process is typically not directly accessible, but is approximated by a set of measurable quantities in the spirit of  $C(X)$  (but admittedly often much smaller than  $C(X)$ ). And moreover, by specifying  $\mathbb{E}(f(\eta_t))$  for all  $f \in C(X)$  we have completely characterized the distribution of the process at time  $t$ , since  $C(X)$  is dual to the set  $\mathcal{M}_1(X)$  of all probability measures on  $X$ .<sup>1</sup>

**Definition 1.5** For a given process  $(\eta_t : t \geq 0)$  on  $X$ , for each  $t \geq 0$  we define the operator

$$S(t) : C(X) \rightarrow C(X) \quad \text{by} \quad (S(t)f)(\zeta) := \mathbb{E}^\zeta f(\eta_t). \quad (1.35)$$

In general  $f \in C(X)$  does not imply  $S(t)f \in C(X)$ , but all the processes we consider have this property and are called *Feller processes*.

**Proposition 1.3** Let  $(\eta_t : t \geq 0)$  be a Feller process on  $X$ . Then the family  $(S(t) : t \geq 0)$  is a *Markov semigroup*, i.e.

- (a)  $S(0) = Id$ , (identity at  $t = 0$ )
- (b)  $t \mapsto S(t)f$  is right-continuous for all  $f \in C(X)$ , (right-continuity)
- (c)  $S(t+s)f = S(t)S(s)f$  for all  $f \in C(X)$ ,  $s, t \geq 0$ , (semigroup/Markov property)
- (d)  $S(t)1 = 1$  for all  $t \geq 0$ , (conservation of probability)
- (e)  $S(t)f \geq 0$  for all non-negative  $f \in C(X)$ . (positivity)

**Proof.** (a)  $S(0)f(\zeta) = \mathbb{E}^\zeta(f(\eta_0)) = f(\zeta)$  since  $\eta_0 = \zeta$  which is equivalent to (a) of Def. 1.1.  
(b) for fixed  $\eta \in X$  right-continuity of  $t \mapsto S(t)f(\eta)$  (a mapping from  $[0, \infty)$  to  $\mathbb{R}$ ) follows directly from right-continuity of  $\eta_t$  and continuity of  $f$ . Right-continuity of  $t \mapsto S(t)f$  (a mapping from  $[0, \infty)$  to  $C(X)$ ) w.r.t. the sup-norm on  $C(X)$  requires to show uniformity in  $\eta$ , which is more involved (see e.g. [12], Chapter IX, Section 1).  
(c) follows from the Markov property of  $\eta_t$  (Def. 1.1(c))

$$\begin{aligned} S(t+s)f(\zeta) &= \mathbb{E}^\zeta f(\eta_{t+s}) = \mathbb{E}^\zeta \left( \mathbb{E}^\zeta (f(\eta_{t+s}) | \mathcal{F}_t) \right) = \mathbb{E}^\zeta \left( \mathbb{E}^{\eta_t} (f(\tilde{\eta}_s)) \right) = \\ &= \mathbb{E}^{\eta_t} ((S(s)f)(\eta_t)) = S(t)S(s)f(\zeta), \end{aligned} \quad (1.36)$$

where  $\tilde{\eta} = \eta_{t+}$  denotes the path of the process started at time  $t$ .

(d)  $S(t)1 = \mathbb{E}^\eta(1) = \mathbb{E}^\eta(\mathbb{1}_{\eta_t}(X)) = 1$  since  $\eta_t \in X$  for all  $t \geq 0$  (conservation of probability).

<sup>1</sup>The fact that probability measures on  $X$  can be characterised by expected values of functions on the dual  $C(X)$  is a direct consequence of the Riesz representation theorem (see e.g. [16], Theorem 2.14).

(e) is immediate by definition.  $\square$

**Remarks.** Note that (b) implies in particular  $S(t)f \rightarrow f$  as  $t \rightarrow 0$  for all  $f \in C(X)$ , which is usually called *strong continuity* of the semigroup (see e.g. [13], Section 19). Furthermore,  $S(t)$  is also *contractive*, i.e. for all  $f \in C(X)$

$$\|S(t)f\|_\infty \leq \|S(t)|f|\|_\infty \leq \|f\|_\infty \|S(t)1\|_\infty = \|f\|_\infty, \quad (1.37)$$

which follows directly from conservation of probability (d). Strong continuity and contractivity imply that  $t \mapsto S(t)f$  is actually uniformly continuous for all  $t > 0$ . Using also the semigroup property (c) we have for all  $t > \epsilon > 0$  and  $f \in C(X)$

$$\|S(t)f - S(t-\epsilon)f\|_\infty = \|S(t-\epsilon)(S(\epsilon)f - f)\|_\infty \leq \|S(\epsilon)f - f\|_\infty, \quad (1.38)$$

which vanishes for  $\epsilon \rightarrow 0$  and implies left-continuity in addition to right-continuity (b).

**Theorem 1.4** *Suppose  $(S(t) : t \geq 0)$  is a Markov semigroup on  $C(X)$ . Then there exists a unique (Feller) Markov process  $(\eta_t : t \geq 0)$  on  $X$  such that*

$$\mathbb{E}^\zeta f(\eta_t) = S(t)f(\zeta) \quad \text{for all } f \in C(X), \zeta \in X \text{ and } t \geq 0. \quad (1.39)$$

**Proof.** see [9] Theorem I.1.5 and references therein

The semigroup  $(S(t) : t \geq 0)$  describes the time evolution of expected values of observables  $f$  on  $X$  for a given Markov process. It provides a full representation of the process which is dual to the path measures  $(\mathbb{P}^\zeta : \zeta \in X)$ .

For a general initial distribution  $\mu \in \mathcal{M}_1(X)$  the path measure (1.5) is  $\mathbb{P}^\mu = \int_X \mathbb{P}^\zeta \mu(d\zeta)$ . Thus

$$\mathbb{E}^\mu f(\eta_t) = \int_X (S(t)f)(\zeta) \mu(d\zeta) = \int_X S(t)f d\mu \quad \text{for all } f \in C(X). \quad (1.40)$$

**Definition 1.6** For a process  $(S(t) : t \geq 0)$  with initial distribution  $\mu$  we denote by  $\mu S(t) \in \mathcal{M}_1(X)$  the *distribution at time  $t$* , which is uniquely determined by

$$\int_X f d[\mu S(t)] := \int_X S(t)f d\mu \quad \text{for all } f \in C(X). \quad (1.41)$$

The notation  $\mu S(t)$  is a convention from functional analysis, where we write

$$\langle S(t)f, \mu \rangle := \int_X S(t)f d\mu = \langle f, S(t)^*\mu \rangle = \langle f, \mu S(t) \rangle. \quad (1.42)$$

The distribution  $\mu$  is in fact evolved by the adjoint operator  $S(t)^*$ , which can also be denoted by  $S(t)^*\mu = \mu S(t)$ . The fact that  $\mu S(t)$  is uniquely specified by (1.41) is again a consequence of the Riesz representation theorem (see e.g. [16], Theorem 2.14).

Since  $(S(t) : t \geq 0)$  has the semigroup structure given in Prop. 1.3(c), in analogy with the proof of Prop. 1.1 we expect that it has the form of an exponential generated by the linearization  $S'(0)$ , i.e.

$${}^{\prime}S(t) = \exp(tS'(0)) = Id + S'(0)t + o(t) \quad \text{with } S(0) = Id, \quad (1.43)$$

which is made precise in the following.

**Definition 1.7** The generator  $\mathcal{L} : D_{\mathcal{L}} \rightarrow C(X)$  for the process  $(S(t) : t \geq 0)$  is given by

$$\mathcal{L}f := \lim_{t \searrow 0} \frac{S(t)f - f}{t} \quad \text{for } f \in D_{\mathcal{L}}, \quad (1.44)$$

where the domain  $D_{\mathcal{L}} \subseteq C(X)$  is the set of functions for which the limit exists.

The limit in (1.44) is to be understood w.r.t. the sup-norm  $\|\cdot\|_{\infty}$  on  $C(X)$ . In general  $D_{\mathcal{L}} \subsetneq C(X)$  is a proper subset for processes on infinite lattices, and we will see later that this is in fact the case even for the simplest examples SEP and CP we introduced above.

**Proposition 1.5**  $\mathcal{L}$  as defined above is a **Markov generator**, i.e.

- (a)  $1 \in D_{\mathcal{L}}$  and  $\mathcal{L}1 = 0$ , (conservation of probability)
- (b) for  $f \in D_{\mathcal{L}}$ ,  $\lambda \geq 0$ :  $\min_{\zeta \in X} f(\zeta) \geq \min_{\zeta \in X} (f - \lambda \mathcal{L}f)(\zeta)$ , (positivity)
- (c)  $D_{\mathcal{L}}$  is dense in  $C(X)$  and the range  $\mathcal{R}(Id - \lambda \mathcal{L}) = C(X)$  for sufficiently small  $\lambda > 0$ .

**Proof.** (a) is immediate from the definition (1.44) and  $S(t)1 = 1$ , the rest is rather technical and can be found in [9] Section I.2 and in references therein.

**Theorem 1.6 (Hille-Yosida)** There is a one-to-one correspondence between Markov generators and semigroups on  $C(X)$ , given by (1.44) and

$$S(t)f := \lim_{n \rightarrow \infty} \left( Id - \frac{t}{n} \mathcal{L} \right)^{-n} f \quad \text{for } f \in C(X), t \geq 0. \quad (1.45)$$

Furthermore, for  $f \in D_{\mathcal{L}}$  also  $S(t)f \in D_{\mathcal{L}}$  for all  $t \geq 0$  and

$$\frac{d}{dt} S(t)f = S(t)\mathcal{L}f = \mathcal{L}S(t)f, \quad (1.46)$$

called the **forward** and **backward equation**, respectively.

**Proof.** See [9], Theorem I.2.9. and references therein.

**Remarks.** Properties (a) and (b) in Prop. 1.5 are related to conservation of probability  $S(t)1 = 1$  and positivity of the semigroup (see Prop. 1.3). By taking closures a linear operator is uniquely determined by its values on a dense set. So property (c) in Prop. 1.5 ensures that the semigroup  $S(t)$  is uniquely defined via (1.45) for all  $f \in C(X)$ , and that  $Id - \frac{t}{n} \mathcal{L}$  is actually invertible for  $n$  large enough, as is required in the definition. The fact that  $D_{\mathcal{L}}$  is dense in  $C(X)$  is basically the statement that  $t \mapsto S(t)$  is indeed differentiable at  $t = 0$ , confirming the intuition (1.43). This can be proved as a consequence of strong continuity of the semigroup. Given that  $S(t)f$  is the unique solution to the backward equation

$$\frac{d}{dt} u(t) = \mathcal{L}u(t) \quad \text{with initial condition } u(0) = f, \quad (1.47)$$

one often writes  $S(t) = e^{t\mathcal{L}}$  in analogy to scalar exponentials as indicated in (1.43).

It can be shown that the  $\mathbb{R}$ -valued process  $f(\eta_t) - S(t)f(\eta_0)$  is a martingale. As an alternative to the Hille-Yosida approach, the process  $(\mathbb{P}^{\zeta} : \zeta \in X)$  can be characterized as a unique solution to

the martingale problem for a given Markov generator  $\mathcal{L}$  (see [9], Sections I.5 and I.6).

**Connection to Markov chains.**

The forward and backward equation, as well as the role of the generator and semigroup are in complete (dual) analogy to the theory of continuous-time Markov chains, where the  $Q$ -matrix generates the time evolution of the distribution at time  $t$  (see e.g. [14] Section 2.1). The approach we introduced above is more general and can of course describe the time evolution of Markov chains with countable  $X$ . With jump rates  $c(\eta, \eta')$  the generator can be computed directly using (1.6) for small  $t \searrow 0$ ,

$$\begin{aligned} S(t)f(\eta) &= \mathbb{E}^\eta(f(\eta_t)) = \sum_{\eta' \in X} \mathbb{P}^\eta(\eta_t = \eta') f(\eta') = \\ &= \sum_{\eta' \neq \eta} c(\eta, \eta') f(\eta') t + f(\eta) \left(1 - \sum_{\eta' \neq \eta} c(\eta, \eta') t\right) + o(t). \end{aligned} \quad (1.48)$$

With the definition (1.44) this yields

$$\mathcal{L}f(\eta) = \lim_{t \searrow 0} \frac{S(t)f - f}{t} = \sum_{\eta' \in X} c(\eta, \eta') (f(\eta') - f(\eta)). \quad (1.49)$$

**Example.** For the simple random walk with state space  $X = \mathbb{Z}$  we have

$$c(\eta, \eta + 1) = p \quad \text{and} \quad c(\eta, \eta - 1) = q, \quad (1.50)$$

while all other transition rates vanish. The generator is given by

$$\mathcal{L}f(\eta) = p(f(\eta + 1) - f(\eta)) + q(f(\eta - 1) - f(\eta)), \quad (1.51)$$

and in the symmetric case  $p = q$  it is proportional to the discrete Laplacian.

In general, since the state space  $X$  for Markov chains is not necessarily compact, we have to restrict ourselves to bounded continuous functions  $f$ . A more detailed discussion of conditions on  $f$  for (1.49) to be a convergent sum for Markov chains can be found in Section 1.6. For IPS with (possibly uncountable)  $X = \{0, 1\}^\Lambda$  we can formally write down similar expressions for the generator. For a lattice gas (e.g. SEP) we have

$$\mathcal{L}f(\eta) = \sum_{x, y \in \Lambda} c(x, y, \eta) (f(\eta^{xy}) - f(\eta)) \quad (1.52)$$

and for pure reaction systems like the CP or the VM

$$\mathcal{L}f(\eta) = \sum_{x \in \Lambda} c(x, \eta) (f(\eta^x) - f(\eta)). \quad (1.53)$$

For infinite lattices  $\Lambda$  convergence of the sums is an issue and we have to find a proper domain  $D_{\mathcal{L}}$  of functions  $f$  for which they are finite.

**Definition 1.8** For  $X = S^\Lambda$  with  $S \subseteq \mathbb{N}$ ,  $f \in C(X)$  is a *cylinder function* if there exists a finite subset  $\Delta_f \subseteq \Lambda$  such that

$$f(\eta) = f(\zeta) \quad \text{for all} \quad \eta, \zeta \in X \quad \text{with} \quad \eta(x) = \zeta(x) \quad \text{for all} \quad x \in \Delta_f, \quad (1.54)$$

i.e.  $f$  depends only on a finite set of coordinates of a configuration. We write  $C_0(X) \subseteq C(X)$  for the set of all cylinder functions.

**Examples.** The indicator function  $\mathbb{1}_\eta$  is in general not a cylinder function (only on finite lattices), whereas the local particle number  $\eta(x)$  or the product  $\eta(x)\eta(x+y)$  are. These functions are important observables, and their expectations correspond to *local densities*

$$\rho(t, x) = \mathbb{E}^\mu(\eta_t(x)) \quad (1.55)$$

and *two-point correlation functions*

$$\rho(t, x, x+y) = \mathbb{E}^\mu(\eta_t(x)\eta_t(x+y)) . \quad (1.56)$$

For  $f \in C_0(X)$  the sum (1.53) contains only finitely many non-zero terms, so converges for any given  $\eta$ . However, we need  $\mathcal{L}f$  to be finite w.r.t. the sup-norm of our Banach space  $(C(X), \|\cdot\|_\infty)$ . To assure this, we also need to impose some regularity conditions on the jump rates. For simplicity we will assume them to be of finite range as explained below. This is much more than is necessary, but it is easy to work with and fulfilled by all the examples we consider. Basically the independence of cylinder functions  $f$  and jump rates  $c$  on coordinates  $x$  outside a finite range  $\Delta \subseteq \Lambda$  can be replaced by a weak dependence on coordinates  $x \notin \Delta$  decaying with increasing  $\Delta$  (see e.g. [9] Sections I.3 and VIII.0 for a more general discussion).

**Definition 1.9** The jump rates of an IPS on  $X = \{0, 1\}^\Lambda$  are said to be of *finite range*  $R > 0$  if for all  $x \in \Lambda$  there exists a finite  $\Delta \subseteq \Lambda$  with  $|\Delta| \leq R$  such that

$$c(x, \eta^z) = c(x, \eta) \quad \text{for all } \eta \in X \text{ and } z \notin \Delta . \quad (1.57)$$

in case of a pure reaction system. For a lattice gas the same should hold for the rates  $c(x, y, \eta)$  for all  $y \in \Lambda$ , with the additional requirement

$$\left| \{y \in \Lambda : c(x, y, \eta) > 0\} \right| \leq R \quad \text{for all } \eta \in X \text{ and } x \in \Lambda . \quad (1.58)$$

**Proposition 1.7** *Under the condition of finite range jump rates,  $\|\mathcal{L}f\|_\infty < \infty$  for all  $f \in C_0(X)$ . Furthermore, the operators  $\mathcal{L}$  defined in (1.52) and (1.53) are uniquely defined by their values on  $C_0(X)$  and are Markov generators in the sense of Prop. 1.5.*

**Proof.** Consider a pure reaction system with rates  $c(x, \eta)$  of finite range  $R$ . Then for each  $x \in \Lambda$ ,  $c(x, \eta)$  assumes only a finite number of values (at most  $2^R$ ), and therefore  $\bar{c}(x) = \sup_{\eta \in X} c(x, \eta) < \infty$ . Then we have for  $f \in C_0(X)$ , depending on coordinates in  $\Delta_f \subseteq \Lambda$ ,

$$\begin{aligned} \|\mathcal{L}f\|_\infty &\leq 2\|f\|_\infty \sup_{\eta \in X} \sum_{x \in \Delta_f} c(x, \eta) \leq 2\|f\|_\infty \sum_{x \in \Delta_f} \sup_{\eta \in X} c(x, \eta) \leq \\ &\leq 2\|f\|_\infty \sum_{x \in \Delta_f} \bar{c}(x) < \infty , \end{aligned} \quad (1.59)$$

since the last sum is finite with finite summands. A similar computation works for lattice gases. The proof of the second statement is more involved, see e.g. [9], Theorem I.3.9. Among many other points, this involves choosing a 'right' metric such that  $C_0(X)$  is dense in  $C(X)$ , which is not the case for the one induced by the sup-norm.  $\square$

Generators are linear operators and Prop. 1.5 then implies that the sum of two or more generators is again a Markov generator (modulo technicalities regarding domains, which can be substantial in more general situations than ours, see e.g. [13]). In that way we can define more general

processes, e.g. a sum of (1.52) and (1.53) could define a contact process with nearest-neighbour particle motion. In general such mixed processes are called *reaction-diffusion processes* and are extremely important in applications e.g. in chemistry or material science [33]. They will not be covered in these notes where we concentrate on developing the mathematical theory for the most basic models.

## 1.5 Stationary measures and reversibility

**Definition 1.10** A measure  $\mu \in \mathcal{M}_1(X)$  is *stationary* or *invariant* if  $\mu S(t) = \mu$  or, equivalently,

$$\int_X S(t)f d\mu = \int_X f d\mu \quad \text{or shorter} \quad \mu(S(t)f) = \mu(f) \quad \text{for all } f \in C(X). \quad (1.60)$$

The set of all invariant measures of a process is denoted by  $\mathcal{I}$ . A measure  $\mu$  is called *reversible* if

$$\mu(fS(t)g) = \mu(gS(t)f) \quad \text{for all } f, g \in C(X). \quad (1.61)$$

To simplify notation here and in the following we use the standard notation  $\mu(f) = \int_X f d\mu$  for integration. This is also the expected value w.r.t. the measure  $\mu$ , but we use the symbol  $\mathbb{E}$  only for expectations on path space w.r.t. the measure  $\mathbb{P}$ .

Taking  $g = 1$  in (1.61) we see that every reversible measure is also stationary. Stationarity of  $\mu$  implies that

$$\mathbb{P}^\mu(\eta_t \in A) = \mathbb{P}^\mu(\eta_{t+} \in A) \quad \text{for all } t \geq 0, A \in \mathcal{F}, \quad (1.62)$$

using the Markov property (Def. 1.1(c)) with notation (1.5) and (1.60). Using  $\eta_t \sim \mu$  as initial distribution, the definition of a stationary process can be extended to negative times on the path space  $D(-\infty, \infty)$ . If  $\mu$  is also reversible, this implies

$$\mathbb{P}^\mu(\eta_{t+} \in A) = \mathbb{P}^\mu(\eta_{t-} \in A) \quad \text{for all } t \geq 0, A \in \mathcal{F}, \quad (1.63)$$

i.e. the process is time-reversible. More details on this are given at the end of this section.

**Proposition 1.8** Consider a Feller process on a compact state space  $X$  with generator  $\mathcal{L}$ . Then

$$\mu \in \mathcal{I} \quad \Leftrightarrow \quad \mu(\mathcal{L}f) = 0 \quad \text{for all } f \in C_0(X), \quad (1.64)$$

and similarly

$$\mu \text{ is reversible} \quad \Leftrightarrow \quad \mu(f\mathcal{L}g) = \mu(g\mathcal{L}f) \quad \text{for all } f, g \in C_0(X). \quad (1.65)$$

**Proof.** The correspondence between semigroups and generators in the is given Hille-Yosida theorem in terms of limits in (1.44) and (1.45). By strong continuity of  $S(t)$  in  $t = 0$  and restricting to  $f \in C_0(X)$  we can re-write both conditions as

$$\mathcal{L}f := \lim_{n \rightarrow \infty} \underbrace{\frac{S(1/n)f - f}{1/n}}_{:=g_n} \quad \text{and} \quad S(t)f := \lim_{n \rightarrow \infty} \underbrace{\left( Id + \frac{t}{n} \mathcal{L} \right)^n f}_{:=h_n}. \quad (1.66)$$

Now  $\mu \in \mathcal{I}$  implies that for all  $n \in \mathbb{N}$

$$\mu(S(1/n)f) = \mu(f) \quad \Rightarrow \quad \mu(g_n) = 0. \quad (1.67)$$

Then we have

$$\mu(\mathcal{L}f) = \mu\left(\lim_{n \rightarrow \infty} g_n\right) = \lim_{n \rightarrow \infty} \mu(g_n) = 0, \quad (1.68)$$

by bounded (or dominated) convergence, since  $g_n$  converges in  $(C(X), \|\cdot\|_\infty)$  as long as  $f \in C_0(X)$ ,  $X$  is compact and  $\mu(X) = 1$ .

On the other hand, if  $\mu(\mathcal{L}f) = 0$  for all  $f \in C_0(X)$ , we have by linearity

$$\mu(h_n) = \mu\left(\left(\text{Id} + \frac{t}{n}\mathcal{L}\right)^n f\right) = \sum_{k=0}^n \binom{n}{k} \frac{t^k}{n^k} \mu(\mathcal{L}^k f) = \mu(f) \quad (1.69)$$

using the binomial expansion, where only the term with  $k = 0$  contributes with  $\mathcal{L}^0 = \text{Id}$ . This is by assumption since  $\mu(\mathcal{L}^k f) = \mu(\mathcal{L}(\mathcal{L}^{k-1} f)) = 0$  and  $\mathcal{L}^{k-1} f \in C_0(X)$ . Then the same limit argument as above (1.68) implies  $\mu(S(t)f) = \mu(f)$ .

This finishes the proof of (1.64), a completely analogous argument works for the equivalence (1.65) on reversibility.  $\square$

It is well known for Markov chains that on a finite state space there exists at least one stationary distribution (see Section 1.6). For IPS compactness of the state spaces  $X$  ensures a similar result.

**Theorem 1.9** *For every Feller process with compact state space  $X$  we have:*

(a)  $\mathcal{I}$  is non-empty, compact and convex.

(b) Suppose the weak limit  $\mu = \lim_{t \rightarrow \infty} \pi S(t)$  exists for some initial distribution  $\pi \in \mathcal{M}_1(X)$ , i.e.

$$\pi S(t)(f) = \int_X S(t)f d\pi \rightarrow \mu(f) \quad \text{for all } f \in C(X), \quad (1.70)$$

then  $\mu \in \mathcal{I}$ .

**Proof.** (a) Convexity of  $\mathcal{I}$  follows directly from two basic facts. Firstly, a convex combination of two probability measures  $\mu_1, \mu_2 \in \mathcal{M}_1(X)$  is again a probability measure, i.e.

$$\nu := \lambda\mu_1 + (1 - \lambda)\mu_2 \in \mathcal{M}_1(X) \quad \text{for all } \lambda \in [0, 1]. \quad (1.71)$$

Secondly, the stationarity condition (1.64) is linear, i.e. if  $\mu_1, \mu_2 \in \mathcal{I}$  then so is  $\nu$  since

$$\nu(\mathcal{L}f) = \lambda\mu_1(\mathcal{L}f) + (1 - \lambda)\mu_2(\mathcal{L}f) = 0 \quad \text{for all } f \in C(X). \quad (1.72)$$

$\mathcal{I}$  is a closed subset of  $\mathcal{M}_1(X)$  if we have

$$\mu_1, \mu_2, \dots \in \mathcal{I}, \quad \mu_n \rightarrow \mu \text{ weakly, implies } \mu \in \mathcal{I}. \quad (1.73)$$

But this is immediate by weak convergence, since for all  $f \in C(X)$

$$\mu_n(\mathcal{L}f) = 0 \quad \text{for all } n \in \mathbb{N} \quad \Rightarrow \quad \mu(\mathcal{L}f) = \lim_{n \rightarrow \infty} \mu_n(\mathcal{L}f) = 0. \quad (1.74)$$

Under the topology of weak convergence  $\mathcal{M}_1(X)$  is compact since  $X$  is compact<sup>1</sup>, and therefore also  $\mathcal{I} \subseteq \mathcal{M}_1(X)$  is compact since it is a closed subset of a convex set.

<sup>1</sup>For more details on weak convergence see e.g. [19], Section 2.

Non-emptiness: By compactness of  $\mathcal{M}_1(X)$  there exists a convergent subsequence of  $\pi S(t)$  for every  $\pi \in \mathcal{M}_1(X)$ . With (b) the limit is in  $\mathcal{I}$ .

(b) Let  $\mu := \lim_{t \rightarrow \infty} \pi S(t)$ . Then  $\mu \in \mathcal{I}$  since for all  $f \in C(X)$ ,

$$\begin{aligned} \mu(S(s)f) &= \lim_{t \rightarrow \infty} \int_X S(s)f d[\pi S(t)] = \lim_{t \rightarrow \infty} \int_X S(t)S(s)f d\pi = \\ &= \lim_{t \rightarrow \infty} \int_X S(t+s)f d\pi = \lim_{t \rightarrow \infty} \int_X S(t)f d\pi = \\ &= \lim_{t \rightarrow \infty} \int_X f d[\pi S(t)] = \mu(f). \end{aligned} \tag{1.75}$$

□

**Remark.** By the Krein Milman theorem (see e.g. [17], Theorem 3.23), compactness and convexity of  $\mathcal{I} \subseteq \mathcal{M}_1(X)$  implies that  $\mathcal{I}$  is the closed convex hull of its extreme points  $\mathcal{I}_e$ , which are called **extremal invariant measures**. Every invariant measure can therefore be written as a convex combination of members of  $\mathcal{I}_e$ , so the extremal measures are the ones we need to find for a given process.

**Definition 1.11** A Markov process with semigroup  $(S(t) : t \geq 0)$  is *ergodic* if

- (a)  $\mathcal{I} = \{\mu\}$  is a singleton, and (unique stationary measure)
- (b)  $\lim_{t \rightarrow \infty} \pi S(t) = \mu$  for all  $\pi \in \mathcal{M}_1(X)$ . (convergence to equilibrium)

Phase transitions are related to the breakdown of ergodicity and in particular to non-uniqueness of stationary measures. This can be the result of the presence of absorbing states (e.g. CP), or of spontaneous symmetry breaking/breaking of conservation laws (e.g. SEP or VM) as is discussed later. On finite lattices, IPS are Markov chains which are known to have a unique stationary distribution under reasonable assumptions of non-degeneracy (see Section 1.6). Therefore, mathematically phase transitions occur only in infinite systems. Infinite systems are often interpreted/studied as limits of finite systems, which show traces of a phase transition by divergence or non-analytic behaviour of certain observables. In terms of applications, infinite systems are approximations or idealizations of real systems which may be large but are always finite, so results have to be interpreted with care.

There is a well developed mathematical theory of phase transitions for reversible systems provided by the framework of Gibbs measures (see e.g. [10]). But for IPS which are in general non-reversible, the notion of phase transitions is not unambiguous, and we will try to get an understanding by looking at several examples.

**Further remarks on reversibility.**

We have seen before that a stationary process can be extended to negative times on the path space  $D(-\infty, \infty)$ . A time reversed stationary process is again a stationary Markov process and the time evolution is given by adjoint operators as explained in the following.

Let  $\mu \in \mathcal{M}_1(X)$  be the stationary measure of the process  $(S(t) : t \geq 0)$  and consider

$$L^2(X, \mu) = \{f \in C(X) : \mu(f^2) < \infty\} \tag{1.76}$$

the set of test functions square integrable w.r.t.  $\mu$ . With the inner product  $\langle f, g \rangle = \mu(fg)$  the closure of this (w.r.t. the metric given by the inner product) is a Hilbert space, and the generator

$\mathcal{L}$  and the  $S(t)$ ,  $t \geq 0$  are bounded linear operators on  $L^2(X, \mu)$ . They are uniquely defined by their values on  $C(X)$ , which is a dense subset of the closure of  $L^2(X, \mu)$ . Therefore they have an *adjoint* operator  $\mathcal{L}^*$  and  $S(t)^*$ , respectively, uniquely defined by

$$\langle S(t)^* f, g \rangle = \mu(g S(t)^* f) = \mu(f S(t) g) = \langle f, S(t) g \rangle \quad \text{for all } f, g \in L^2(X, \mu), \quad (1.77)$$

and analogously for  $\mathcal{L}^*$ . Note that the adjoint operators on the self-dual Hilbert space  $L^2(X, \mu)$  are not the same as the adjoints mentioned in (1.42) on  $\mathcal{M}_1(X)$  (dual to  $C(X)$ ), which evolve the probability measures. To compute the action of the adjoint operator note that for all  $g \in L^2(X, \mu)$

$$\begin{aligned} \mu(g S(t)^* f) &= \int_X f S(t) g \, d\mu = \mathbb{E}^\mu(f(\eta_0) g(\eta_t)) = \mathbb{E}^\mu\left(\mathbb{E}(f(\eta_0)|\eta_t) g(\eta_t)\right) = \\ &= \int_X \mathbb{E}(f(\eta_0)|\eta_t = \zeta) g(\zeta) \mu(d\zeta) = \mu\left(g \mathbb{E}(f(\eta_0)|\eta_t = \cdot)\right), \end{aligned} \quad (1.78)$$

where the identity between the first and second line is due to  $\mu$  being the stationary measure. Since this holds for all  $g$  it implies that

$$S(t)^* f(\eta) = \mathbb{E}(f(\eta_0)|\eta_t = \eta), \quad (1.79)$$

so the adjoint operator describes the evolution of the time-reversed process. Similarly, it can be shown that the adjoint generator  $\mathcal{L}^*$  is actually the generator of the adjoint semigroup  $S(t)^* : t \geq 0$ . This includes some technicalities with domains of definition, see e.g. [18] and references therein. The process is time-reversible if  $\mathcal{L} = \mathcal{L}^*$  and therefore reversibility is equivalent to  $\mathcal{L}$  and  $S(t)$  being *self-adjoint* as in (1.61) and (1.65).

## 1.6 Simplified theory for Markov chains

For Markov chains the state space  $X$  is countable, but not necessarily compact, think e.g. of a random walk on  $X = \mathbb{Z}$ . Therefore we have to restrict the construction of the semigroups to bounded continuous functions

$$C_b(X) := \{f : X \rightarrow \mathbb{R} \text{ continuous and bounded}\}. \quad (1.80)$$

In particular cases a larger space could be used, but the set  $C_b(X)$  of bounded observables is sufficient to uniquely characterize the distribution of the of the Markov chain<sup>1</sup>. Note that if  $X$  is compact (e.g. for finite state Markov chains or for all IPS considered in Section 1.4), then  $C_b(X) = C(X)$ . The domain of the generator (1.49)

$$\mathcal{L}f(\eta) = \sum_{\eta' \neq \eta} c(\eta, \eta') (f(\eta') - f(\eta)) \quad (1.81)$$

for a Markov chain is then given by the full set of observables  $\mathcal{D}_{\mathcal{L}} = C_b(X)$ . This follows from the uniform bound  $c_\eta \leq \bar{c}$  (1.9) on the jump rates, since for every  $f \in C_b(X)$

$$\|\mathcal{L}f\|_\infty = \sup_{\eta \in X} \mathcal{L}f(\eta) \leq 2\|f\|_\infty \sup_{\eta \in X} \sum_{\eta' \in X} c(\eta, \eta') = 2\|f\|_\infty \sup_{\eta \in X} c_\eta < \infty. \quad (1.82)$$

---

<sup>1</sup>cf. weak convergence of distributions, which is usually defined via expected values of  $f \in C_b(X)$  (see e.g. [13], Chapter 4).

In particular, indicator functions  $f = \mathbb{1}_\eta : X \rightarrow \{0, 1\}$  are always in  $C_b(X)$  and we have

$$\int_X S(t) f d\mu = [\mu S(t)](\eta) =: p_t(\eta) \quad (1.83)$$

for the distribution at time  $t$  with  $p_0(\eta) = \mu(\eta)$ . Using this and (1.81) we get for the right-hand side of the backward equation (1.47) for all  $\eta \in X$

$$\begin{aligned} \int_X \mathcal{L}S(t) \mathbb{1}_\eta d\mu &= \sum_{\zeta \in X} \mu(\zeta) \sum_{\zeta' \in X} c(\zeta, \zeta') (S(t) \mathbb{1}_\eta(\zeta') - S(t) \mathbb{1}_\eta(\zeta)) = \\ &= \sum_{\zeta \in X} [\mu S(t)](\zeta) \left( c(\zeta, \eta) - \mathbb{1}_\eta(\zeta) \sum_{\zeta' \in X} c(\zeta, \zeta') \right) = \\ &= \sum_{\zeta \in X} p_t(\zeta) c(\zeta, \eta) - p_t(\eta) \sum_{\zeta' \in X} c(\eta, \zeta'), \end{aligned} \quad (1.84)$$

where we use the convention  $c(\zeta, \zeta) = 0$  for all  $\zeta \in X$ . In summary we get

$$\frac{d}{dt} p_t(\eta) = \sum_{\eta' \neq \eta} \left( p_t(\eta') c(\eta', \eta) - p_t(\eta) c(\eta, \eta') \right), \quad p_0(\eta) = \mu(\eta). \quad (1.85)$$

This is called the *master equation*, with intuitive gain and loss terms into state  $\eta$  on the right-hand side. It makes sense only for countable  $X$ , and in that case it is actually equivalent to (1.47), since the indicator functions form a basis of  $C_b(X)$ .

Analogous to the master equation (and using the same notation), we can get a meaningful relation for Markov chains by inserting the indicator function  $f = \mathbb{1}_\eta$  in the stationarity condition (1.64). This yields with (1.81)

$$\mu(\mathcal{L} \mathbb{1}_\eta) = \sum_{\eta' \neq \eta} \left( \mu(\eta') c(\eta', \eta) - \mu(\eta) c(\eta, \eta') \right) = 0 \quad \text{for all } \eta \in X, \quad (1.86)$$

so that  $\mu$  is a stationary solution of the master equation (1.85). A short computation yields

$$\mu(\mathbb{1}_\eta \mathcal{L} \mathbb{1}_{\eta'}) = \sum_{\zeta \in X} \mu(\zeta) \mathbb{1}_\eta(\zeta) \sum_{\xi \in X} c(\zeta, \xi) (\mathbb{1}_{\eta'}(\xi) - \mathbb{1}_{\eta'}(\zeta)) = \mu(\eta) c(\eta, \eta'), \quad (1.87)$$

again using  $c(\zeta, \zeta) = 0$  for all  $\zeta \in X$ . So inserting  $f = \mathbb{1}_\eta$  and  $g = \mathbb{1}_{\eta'}$  for  $\eta' \neq \eta$  into the reversibility condition (1.65) on both sides we get

$$\mu(\eta') c(\eta', \eta) = \mu(\eta) c(\eta, \eta') \quad \text{for all } \eta, \eta' \in X, \eta \neq \eta', \quad (1.88)$$

which are called *detailed balance relations*. So if  $\mu$  is reversible, every individual term in the sum (1.86) vanishes. On the other hand, not every solution of (1.86) has to fulfill (1.88), i.e. there are stationary measures which are not reversible. The detailed balance equations are typically easy to solve for  $\mu$ , so if reversible measures exist they can be found as solutions of (1.88).

**Examples.** Consider the simple random walk on the torus  $X = \mathbb{Z}/L\mathbb{Z}$ , moving with rate  $p$  to the right and  $q$  to the left. The uniform measure  $\mu(\eta) = 1/L$  is an obvious solution to the stationary master equation (1.86). However, the detailed balance relations are only fulfilled in the symmetric case  $p = q$ . For the simple random walk on the infinite state space  $X = \mathbb{Z}$  the constant solution

cannot be normalized, and in fact (1.86) does not have a normalized solution.

Another important example is a *birth-death chain* with state space  $X = \mathbb{N}$  and jump rates

$$c(\eta, \eta + 1) = \alpha, \quad c(\eta + 1, \eta) = \beta \quad \text{for all } \eta \in \mathbb{N}. \quad (1.89)$$

In this case the detailed balance relations have the solution

$$\mu(\eta) = (\alpha/\beta)^\eta. \quad (1.90)$$

For  $\alpha < \beta$  this can be normalized, yielding a stationary, reversible measure for the process.

In particular not every Markov chain has a stationary distribution. If  $X$  is finite there exists at least one stationary distribution, as a direct result of the Perron-Frobenius theorem in linear algebra. For general countable (possibly infinite) state space  $X$ , existence of a stationary measure is equivalent to positive recurrence of the Markov chain (cf. [14], Section 3.5).

What about uniqueness of stationary distributions?

**Definition 1.12** A Markov chain  $(\mathbb{P}^\eta : \eta \in X)$  is called *irreducible*, if for all  $\eta, \eta' \in X$

$$\mathbb{P}^\eta(\eta_t = \eta') > 0 \quad \text{for some } t \geq 0. \quad (1.91)$$

So an irreducible Markov chain can sample the whole state space, and it can be shown that this implies that it has at most one stationary distribution (cf. [14], Section 3.5). For us most important is the following statement on ergodicity as defined in Def. 1.11.

**Proposition 1.10** *An irreducible Markov chain with finite state space  $X$  is ergodic.*

**Proof.** Again a result of linear algebra, in particular the Perron-Frobenius theorem: The generator can be understood as a finite matrix  $c(\eta, \eta')$ , which has eigenvalue 0 with unique eigenvector  $\mu$ . All other eigenvalues  $\lambda_i$  have negative real part, and the so-called *spectral gap*

$$\gamma := -\inf_i \operatorname{Re}(\lambda_i) \quad (1.92)$$

determines the speed of convergence to equilibrium. For every initial distribution  $\pi \in \mathcal{M}_1(X)$  we have weak convergence with

$$|\pi S(t)(f) - \mu(f)| \leq C e^{-\gamma t} \quad \text{for all } f \in C(X). \quad (1.93)$$

□

The spectrum of the generator plays a similar role also for general Markov processes and IPS. The spectral gap is often hard to calculate, useful estimates can be found for reversible processes (see e.g. [11], Appendix 3 and also [18]).

## 2 The asymmetric simple exclusion process

As given in Def. 1.2 an *exclusion process (EP)* has state space  $X = \{0, 1\}^\Lambda$  on a lattice  $\Lambda$ . The process is characterized by the generator

$$\mathcal{L}f(\eta) = \sum_{x,y \in \Lambda} c(x,y,\eta) (f(\eta^{xy}) - f(\eta)) \quad (2.1)$$

with jump rates

$$c(x,y,\eta) = p(x,y) \eta(x) (1 - \eta(y)) . \quad (2.2)$$

$p(x,y)$  are irreducible transition rates of a single random walker on  $\Lambda$ . For the simple EP (SEP)  $\Lambda$  is a regular lattice such as  $\mathbb{Z}^d$  and  $p(x,y) = 0$  whenever  $x$  and  $y$  are not nearest neighbours. In this chapter we focus on results and techniques that apply to the asymmetric SEP (ASEP) as well as to the symmetric SEP (SSEP). For the latter there are more detailed results available based on reversibility of the process (see e.g. [9], Section VIII.1).

### 2.1 Stationary measures and conserved quantities

**Definition 2.1** For a function  $\rho : \Lambda \rightarrow [0, 1]$ ,  $\nu_\rho$  is a *product measure* on  $X$  if for all  $k \in \mathbb{N}$ ,  $x_1, \dots, x_k \in \Lambda$  mutually different and  $n_1, \dots, n_k \in \{0, 1\}$

$$\nu_\rho(\eta(x_1) = n_1, \dots, \eta(x_k) = n_k) = \prod_{i=1}^k \nu_{\rho(x_i)}^1(\eta(x_i) = n_i) , \quad (2.3)$$

where the *single-site marginals* are given by

$$\nu_{\rho(x_i)}^1(\eta(x_i) = 1) = \rho(x_i) \quad \text{and} \quad \nu_{\rho(x_i)}^1(\eta(x_i) = 0) = 1 - \rho(x_i) . \quad (2.4)$$

**Remark.** In other words under  $\nu_\rho$  the  $\eta(x)$  are independent Bernoulli random variables  $\eta(x) \sim Be(\rho(x))$  with *local density*  $\rho(x) = \nu(\eta(x))$ . The above definition can readily be generalized to non-Bernoulli product measures (see e.g. Section 3).

**Theorem 2.1** (a) Suppose  $p(\cdot, \cdot)/C$  is doubly stochastic for some  $C \in (0, \infty)$ , i.e.

$$\sum_{y' \in \Lambda} p(x, y') = \sum_{x' \in \Lambda} p(x', y) = C \quad \text{for all } x, y \in \Lambda , \quad (2.5)$$

then  $\nu_\rho \in \mathcal{I}$  for all constants  $\rho \in [0, 1]$  (*uniform density*).

(b) If  $\lambda : \Lambda \rightarrow [0, \infty)$  fulfills  $\lambda(x) p(x, y) = \lambda(y) p(y, x)$ ,

then  $\nu_\rho \in \mathcal{I}$  with density  $\rho(x) = \frac{\lambda(x)}{1 + \lambda(x)}$ ,  $x \in \Lambda$ .

**Proof.** For stationarity we have to show that  $\nu_\rho(\mathcal{L}f) = 0$  for all  $f \in C_0(X)$ . This condition is linear in  $f$  and every cylinder function can be written as a linear combination of simple functions

$$f_\Delta(\eta) = \begin{cases} 1 & , \eta(x) = 1 \text{ for each } x \in \Delta \\ 0 & , \text{ otherwise} \end{cases} \quad (2.6)$$

for  $\Delta \subseteq \Lambda$  finite<sup>1</sup>. Therefore we have to check the stationarity condition only for such functions where we have

$$\nu_\rho(\mathcal{L}f_\Delta) = \sum_{x,y \in \Lambda} p(x,y) \int_X \eta(x)(1-\eta(y))(f_\Delta(\eta^{xy}) - f_\Delta(\eta)) d\nu_\rho. \quad (2.7)$$

For  $x \neq y$  (we take  $p(x,x) = 0$  for all  $x \in \Lambda$ ) the integral terms in the sum look like

$$\begin{aligned} \int_X f_\Delta(\eta) \eta(x)(1-\eta(y)) d\nu_\rho &= \begin{cases} 0 & , y \in \Delta \\ (1-\rho(y)) \prod_{u \in \Delta \cup \{x\}} \rho(u) & , y \notin \Delta \end{cases} \\ \int_X f_\Delta(\eta^{xy}) \eta(x)(1-\eta(y)) d\nu_\rho &= \begin{cases} 0 & , x \in \Delta \\ (1-\rho(y)) \prod_{u \in \Delta \cup \{x\} \setminus \{y\}} \rho(u) & , x \notin \Delta \end{cases}. \end{aligned} \quad (2.8)$$

This follows from the fact that the integrands take values only in  $\{0, 1\}$  and the right-hand side is therefore the probability of the integrand being 1. Then re-arranging the sum we get

$$\nu_\rho(\mathcal{L}f_\Delta) = \sum_{\substack{x \in A \\ y \notin A}} \left[ \rho(y)(1-\rho(x))p(y,x) - \rho(x)(1-\rho(y))p(x,y) \right] \prod_{u \in A \setminus \{x\}} \rho(u). \quad (2.9)$$

Assumption of (b) is equivalent to

$$\frac{\rho(x)}{1-\rho(x)} p(x,y) = \frac{\rho(y)}{1-\rho(y)} p(y,x), \quad (2.10)$$

so the square bracket vanishes for all  $x, y$  in the sum (2.9). For  $\rho(x) \equiv \rho$  in (a) we get

$$\nu_\rho(\mathcal{L}f_\Delta) = \rho^{|\Delta|} (1-\rho) \sum_{\substack{x \in \Delta \\ y \notin \Delta}} [p(y,x) - p(x,y)] = 0 \quad (2.11)$$

due to  $p(\cdot, \cdot)$  being proportional to a doubly-stochastic. □

For the ASEP (1.28) in one dimension with  $\Lambda = \mathbb{Z}$  we have:

- Theorem 2.1(a) holds with  $C = p + q$  and therefore  $\nu_\rho \in \mathcal{I}$  for all  $\rho \in [0, 1]$ . These measures have homogeneous density; they are reversible iff  $p = q$ , which is immediate from time-reversibility.
- Also Theorem 2.1(b) is fulfilled with  $\lambda(x) = c(p/q)^x$  for all  $c \geq 0$ , since  $c(p/q)^x p = c(p/q)^{x+1} q$ . Therefore

$$\nu_\rho \in \mathcal{I} \quad \text{with} \quad \rho(x) = \frac{c(p/q)^x}{1 + c(p/q)^x} \quad \text{for all } c \geq 0. \quad (2.12)$$

For  $p = q$  these measures are homogeneous and in fact the same ones we found above using Theorem 2.1(a). For  $p \neq q$  the measures are not homogeneous and since e.g. for  $p > q$

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<sup>1</sup>Remember that cylinder functions depend only on finitely many coordinates and with local state space  $\{0, 1\}$  therefore only take finitely many different values.

the density of particles (holes) is exponentially decaying as  $x \rightarrow \pm\infty$  they concentrate on configurations such that

$$\sum_{x<0} \eta(x) < \infty \quad \text{and} \quad \sum_{x \geq 0} (1 - \eta(x)) < \infty. \quad (2.13)$$

These are called *blocking measures* and turn out to be reversible also for  $p \neq q$  (see [20]). Note that these measures are not translation invariant, but the dynamics of the ASEP is.

- To further understand the family of blocking measures, note that there are only countably many configurations with property (2.13), forming the disjoint union of

$$X_n = \left\{ \eta : \sum_{x<n} \eta(x) = \sum_{x \geq n} (1 - \eta(x)) < \infty \right\}, \quad n \in \mathbb{Z}. \quad (2.14)$$

Whenever a particle crosses the bond  $(n-1, n)$  a hole crosses in the other direction, so the process cannot leave  $X_n$  and it is an invariant set for the ASEP. This is of course a consequence of the fact that no particles are created or destroyed. Conditioned on  $X_n$  which is countable, the ASEP is an irreducible MC with unique stationary distribution  $\nu_n := \nu_\rho(\cdot | X_n)$ . Due to conditioning on  $X_n$  the distribution  $\nu_n$  does actually not depend on  $\rho$  any more (cf. next section for a more detailed discussion). In [20] Liggett showed using couplings that all extremal stationary measures of the ASEP in one dimension are

$$\mathcal{I}_e = \{ \nu_\rho : \rho \in [0, 1] \} \cup \{ \nu_n : n \in \mathbb{Z} \}. \quad (2.15)$$

To stress the role of the boundary conditions let us consider another example. For the ASEP on a one-dimensional torus  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  we have:

- Theorem 2.1(a) still applies so  $\nu_\rho \in \mathcal{I}$  for all  $\rho \in [0, 1]$ . But part (b) does no longer hold due to periodic boundary conditions, so there are no blocking measures.

Under  $\nu_\rho$  the total number of particles in the system is a binomial random variable

$$\Sigma_L(\eta) := \sum_{x \in \Lambda} \eta(x) \sim Bi(L, \rho) \quad \text{where} \quad \nu_\rho(\Sigma_L = N) = \binom{L}{N} \rho^N (1-\rho)^{L-N} \quad (2.16)$$

Originating from statistical mechanics, the measures  $\{ \nu_\rho : \rho \in [0, 1] \}$  for the finite lattice  $\Lambda_L$  are called *grand-canonical measures/ensemble*.

- If we fix the number of particles at time 0, i.e.  $\Sigma_L(\eta_0) = N$ , we condition the ASEP on

$$X_{L,N} = \{ \eta : \Sigma_L(\eta) = N \} \subsetneq X_L, \quad (2.17)$$

which is an invariant set since the number of particles is conserved by the dynamics. For each  $N \in \mathbb{N}$ , the process is irreducible on  $X_{L,N}$  and  $|X_{L,N}| = \binom{L}{N}$  is finite. Therefore it has a unique stationary measure  $\pi_{L,N}$  on  $X_{L,N}$  and the  $\{ \pi_{L,N} : N = 0, \dots, L \}$  are called *canonical measures/ensemble*.

## 2.2 Symmetries and conservation laws

**Definition 2.2** For a given Feller process  $(S(t) : t \geq 0)$  a bounded<sup>1</sup> linear operator  $T : C(X) \rightarrow C(X)$  is called a *symmetry*, if it commutes with the semigroup. So for all  $t \geq 0$  we have  $S(t)T = TS(t)$ , i.e.

$$S(t)(Tf)(\eta) = T(S(t)f)(\eta), \quad \text{for all } f \in C(X), \eta \in X. \quad (2.18)$$

**Proposition 2.2** For a Feller process with generator  $\mathcal{L}$ , a bounded linear operator  $T : C(X) \rightarrow C(X)$  is a symmetry iff  $\mathcal{L}T = T\mathcal{L}$ , i.e.

$$\mathcal{L}(Tf)(\eta) = T(\mathcal{L}f)(\eta), \quad \text{for all } f \in C_0(X). \quad (2.19)$$

We denote the set of all symmetries by  $\mathcal{S}(\mathcal{L})$  or simply  $\mathcal{S}$ . The symmetries form a semigroup w.r.t. composition, i.e.

$$T_1, T_2 \in \mathcal{S} \quad \Rightarrow \quad T_1T_2 = T_1 \circ T_2 \in \mathcal{S}. \quad (2.20)$$

**Proof.** The first part is similar to the proof of Prop. 1.8 on stationarity (see problem sheet). For the second part, note that composition of operators is associative. Then for  $T_1, T_2 \in \mathcal{S}$  we have

$$\mathcal{L}(T_1T_2) = (\mathcal{L}T_1)T_2 = (T_1\mathcal{L})T_2 = T_1(\mathcal{L}T_2) = (T_1T_2)\mathcal{L} \quad (2.21)$$

so that  $T_1T_2 \in \mathcal{S}$ . □

**Proposition 2.3** For a bijection  $\tau : X \rightarrow X$  let  $Tf := f \circ \tau$ , i.e.  $Tf(\eta) = f(\tau\eta)$  for all  $\eta \in X$ . Then  $T$  is a symmetry for the process  $(S(t) : t \geq 0)$  iff

$$S(t)(f \circ \tau) = (S(t)f) \circ \tau \quad \text{for all } f \in C(X). \quad (2.22)$$

Such  $T$  (or equivalently  $\tau$ ) are called **simple symmetries**. Simple symmetries are invertible and form a group.

**Proof.** The first statement is immediate by the definition,  $T$  is bounded since  $\|f \circ \tau\|_\infty = \|f\|_\infty$  and obviously linear.

In general compositions of symmetries are symmetries according to Prop. 2.2, and if  $\tau_1, \tau_2 : X \rightarrow X$  are simple symmetries then the composition  $\tau_1 \circ \tau_2 : X \rightarrow X$  is also a simple symmetry. A simple symmetry  $\tau$  is a bijection, so it has an inverse  $\tau^{-1}$ . Then we have for all  $f \in C(X)$  and all  $t \geq 0$

$$(S(t)(f \circ \tau^{-1})) \circ \tau = S(t)(f \circ \tau^{-1} \circ \tau) = S(t)f \quad (2.23)$$

since  $\tau \in \mathcal{S}$ . Composing with  $\tau^{-1}$  leads to

$$(S(t)(f \circ \tau^{-1})) \circ \tau \circ \tau^{-1} = S(t)(f \circ \tau^{-1}) = (S(t)f) \circ \tau^{-1}, \quad (2.24)$$

so that  $\tau^{-1}$  is also a simple symmetry. □

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<sup>1</sup> $T : C(X) \rightarrow C(X)$  is bounded if there exists  $B > 0$  such that for all  $f \in C(X)$ ,  $\|f \circ \tau\|_\infty \leq B\|f\|_\infty$ .

**Example.** For the ASEP on  $\Lambda = \mathbb{Z}$  the translations  $\tau_x : X \rightarrow X$  for  $x \in \Lambda$ , defined by

$$(\tau_x \eta)(y) = \eta(y - x) \quad \text{for all } y \in \Lambda \quad (2.25)$$

are simple symmetries. This can be easily seen since the jump rates are invariant under translations, i.e. we have for all  $x, y \in \Lambda$

$$\begin{aligned} c(x, x+1, \eta) &= p\eta(x)(1 - \eta(x+1)) = p\eta(x+y-y)(1 - \eta(x+1+y-y)) = \\ &= c(x+y, x+1+y, \tau_y \eta). \end{aligned} \quad (2.26)$$

An analogous relation holds for jumps to the left with rate  $c(x, x-1, \eta) = q\eta(x)(1 - \eta(x-1))$ . Note that the family  $\{\tau_x : x \in \Lambda\}$  forms a group. The same symmetry holds for the ASEP on  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  with periodic boundary conditions, where there are only  $L$  distinct translations  $\tau_x$  for  $x = 0, \dots, L-1$  (since e.g.  $\tau_L = \tau_0$  etc.). The argument using symmetry of the jump rates can be made more general.

**Proposition 2.4** *Consider an IPS with jump rates  $c(\eta, \eta')$  in general notation<sup>1</sup>. Then a bijection  $\tau : X \rightarrow X$  is a simple symmetry iff*

$$c(\eta, \eta') = c(\tau\eta, \tau\eta') \quad \text{for all } \eta, \eta' \in X. \quad (2.27)$$

**Proof.** Assuming the symmetry of the jump rates, we have for all  $f \in C_0(X)$  and  $\eta \in X$

$$\begin{aligned} (\mathcal{L}(Tf))(\eta) &= (\mathcal{L}(f \circ \tau))(\eta) = \sum_{\eta' \in X} c(\eta, \eta')(f(\tau\eta') - f(\tau\eta)) = \\ &= \sum_{\eta' \in X} c(\tau\eta, \tau\eta')(f(\tau\eta') - f(\tau\eta)) = \sum_{\zeta' \in X} c(\tau\eta, \zeta')(f(\zeta') - f(\tau\eta)) = \\ &= (\mathcal{L}f)(\tau\eta) = (T(\mathcal{L}f))(\eta), \end{aligned} \quad (2.28)$$

where the identity in the second line just comes from relabeling the sum which is possible since  $\tau$  is bijective and the sum converges absolutely. On the other hand,  $\mathcal{L}T = T\mathcal{L}$  implies that

$$\sum_{\eta' \in X} c(\eta, \eta')(f(\tau\eta') - f(\tau\eta)) = \sum_{\eta' \in X} c(\tau\eta, \tau\eta')(f(\tau\eta') - f(\tau\eta)). \quad (2.29)$$

Since this holds for all  $f \in C_0(X)$  and  $\eta \in X$  it uniquely determines that  $c(\eta, \zeta) = c(\tau\eta, \tau\zeta)$  for all  $\eta, \zeta \in X$  with  $\eta \neq \zeta$ . In fact, if there existed  $\eta, \zeta$  for which this is not the case, we can plug  $f = \mathbb{1}_{\tau\zeta}$  into (2.29) which yields a contradiction. For fixed  $\eta$  both sums then contain only a single term, so this is even possible on infinite lattices even though  $\mathbb{1}_{\tau\zeta}$  is not a cylinder function<sup>2</sup>.  $\square$

**Proposition 2.5** *For an observable  $g \in C(X)$  define the multiplication operator  $T_g := g \text{ Id}$  via*

$$T_g f(\eta) = g(\eta) f(\eta) \quad \text{for all } f \in C(X), \eta \in X. \quad (2.30)$$

*Then  $T_g$  is a symmetry for the process  $(\eta_t : t \geq 0)$  iff  $g(\eta_t) = g(\eta_0)$  for all  $t > 0$ . In that case  $T_g$  (or equivalently  $g$ ) is called a **conservation law** or **conserved quantity**.*

<sup>1</sup>Remember that for fixed  $\eta$  there are only countably many  $c(\eta, \eta') > 0$ .

<sup>2</sup>So the function  $\eta \mapsto \mathcal{L}\mathbb{1}_{\tau\zeta}(\eta)$  would in general not be well defined since it is given by an infinite sum for  $\eta = \tau\zeta$ . But here we are only interested in a single value for  $\eta \neq \zeta$ .

**Proof.** First note that  $T_g$  is linear and bounded since  $\|f\|_\infty \leq \|g\|_\infty \|f\|_\infty$ . If  $g(\eta_t) = g(\eta_0)$  we have for all  $t > 0$ ,  $f \in C(X)$  and  $\eta \in X$

$$(S(t)(T_g f))(\eta) = \mathbb{E}^\eta(g(\eta_t) f(\eta_t)) = g(\eta)(S(t)f)(\eta) = T_g(S(t)f)(\eta). \quad (2.31)$$

On the other hand, if  $T_g$  is a symmetry the above computation implies that for all (fixed)  $t > 0$

$$\mathbb{E}^\eta(g(\eta_t) f(\eta_t)) = \mathbb{E}^\eta(g(\eta) f(\eta)). \quad (2.32)$$

Since this holds for all  $f \in C(X)$  the value of  $g(\eta_t)$  is uniquely specified by the expected values to be  $g(\eta)$  since  $g$  is continuous (cf. argument in (2.29)).  $\square$

**Remarks.** If  $g \in C(X)$  is a conservation law then so is  $h \circ g$  for all  $h : \mathbb{R} \rightarrow \mathbb{R}$  provided that  $h \circ g \in C(X)$ .

A subset  $Y \subseteq X$  is called *invariant* if  $\eta_0 \in Y$  includes  $\eta_t \in Y$  for all  $t > 0$ . Then  $g = \mathbb{1}_Y$  is a conservation law iff  $Y$  is invariant. In general, every level set

$$X_l = \{\eta \in X : g(\eta) = l\} \subseteq X \quad \text{for all } l \in \mathbb{R}, \quad (2.33)$$

for a conserved quantity  $g \in C(X)$  is invariant.

**Examples.** For the ASEP on  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  the number of particles  $\Sigma_L(\eta) = \sum_{x \in \Lambda_L} \eta(x)$  is conserved. The level sets of this integer valued function are the subsets

$$X_{L,N} = \{\eta : \Sigma_L(\eta) = N\} \quad \text{for } N = 0, \dots, L, \quad (2.34)$$

defined in (2.17). In particular the indicator functions  $\mathbb{1}_{X_{L,N}}$  are conserved quantities. Similar conservation laws exist for the ASEP on  $\Lambda = \mathbb{Z}$  in connection with the blocking measures (2.14).

The most important result of this section is the connection between symmetries and stationary measures. For a measure  $\mu$  and a symmetry  $T$  we define the measure  $\mu T$  via

$$(\mu T)(f) = \int_X f d\mu T := \int_X T f d\mu = \mu(T f) \quad \text{for all } f \in C(X), \quad (2.35)$$

analogous to the definition of  $\mu S(t)$  in Def. 1.6.

**Theorem 2.6** For a Feller process  $(S(t) : t \geq 0)$  with state space  $X$  we have

$$\mu \in \mathcal{I}, T \in \mathcal{S} \quad \Rightarrow \quad \frac{1}{\mu T(X)} \mu T \in \mathcal{I}, \quad (2.36)$$

provided that the normalization  $\mu T(X) \in (0, \infty)$ .

**Proof.** For  $\mu \in \mathcal{I}$  and  $T \in \mathcal{S}$  we have for all  $t \geq 0$  and  $f \in C(X)$

$$(\mu T)S(t)(f) = \mu(T S(t)f) = \mu(S(t)T f) \mu S(t)(T f) = \mu(T f) = \mu T(f). \quad (2.37)$$

With  $\mu T(X) \in (0, \infty)$ ,  $\mu T$  can be normalized and  $\frac{1}{\mu T(X)} \mu T \in \mathcal{I}$ .  $\square$

**Remarks.** For  $\mu \in \mathcal{I}$  it will often be the case that  $\mu T = \mu$  so that  $\mu$  is invariant under some  $T \in \mathcal{S}$  and not every symmetry generates a new stationary measure. For ergodic processes  $\mathcal{I} = \{\mu\}$  is a

singleton, so  $\mu$  has to respect all the symmetries of the process, i.e.  $\mu T = \mu$  for all  $T \in \mathcal{S}$ . If  $T_g = g Id$  is a conservation law, then  $\mu T_g = g \mu$ , i.e.

$$\mu T_g(Y) = \int_Y g(\eta) \mu(d\eta) \quad \text{for all measurable } Y \subseteq X. \quad (2.38)$$

So  $g$  is the *density* of  $\mu T_g$  w.r.t.  $\mu$  and one also writes  $g = \frac{d\mu T_g}{d\mu}$ . This implies also that  $\mu T_g$  is *absolutely continuous* w.r.t.  $\mu$  (short  $\mu T_g \ll \mu$ ), which means that for all measurable  $Y$ ,  $\mu(Y) = 0$  implies  $\mu T_g(Y) = 0$ <sup>1</sup>.

For an invariant set  $Y \subseteq X$  and the conservation law  $g = \mathbb{1}_Y$  we have  $\mu T_g = \mathbb{1}_Y \mu$ . If  $\mu(Y) \in (0, \infty)$  the measure of Theorem (2.6) can be written as a *conditional measure*

$$\frac{1}{\mu T_g(X)} \mu T_g = \frac{\mathbb{1}_Y}{\mu(Y)} \mu =: \mu(\cdot | Y) \quad (2.39)$$

concentrating on the set  $Y$ , since the normalization is  $\mu T_g(X) = \mu(\mathbb{1}_Y) = \mu(Y)$ .

**Examples.** The homogeneous product measures  $\nu_\rho$ ,  $\rho \in [0, 1]$  are invariant under the translations  $\tau_x$ ,  $x \in \Lambda$  for all translation invariant lattices with  $\tau_x \Lambda = \Lambda$  such as  $\Lambda = \mathbb{Z}$  or  $\Lambda = \mathbb{Z}/L\mathbb{Z}$ . But the blocking measures  $\nu_n$  for  $\Lambda = \mathbb{Z}$  are not translation invariant, and in fact  $\nu_n = \nu_0 \circ \tau_{-n}$ , so the family of blocking measures is generated from a single one by applying translations.

For  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  we have the invariant sets  $X_{L,N}$  for a fixed number of particles  $N = 0, \dots, L$  as given in (2.17). Since the ASEP is irreducible on  $X_{L,N}$  it has a unique stationary measure  $\pi_{L,N}$  (see previous section). Using the above remark we can write  $\pi_{L,N}$  as a conditional product measure  $\nu_\rho$  (which is also stationary). For all  $\rho \in (0, 1)$  we have (by uniqueness of  $\pi_{L,N}$ )

$$\pi_{L,N} = \nu_\rho(\cdot | X_{L,N}) = \frac{\mathbb{1}_{X_{L,N}}}{\nu_\rho(X_{L,N})} \nu_\rho, \quad (2.40)$$

where  $\nu_\rho(X_{L,N}) = \binom{L}{N} \rho^N (1-\rho)^{L-N}$  is binomial (see previous section). Therefore we can compute explicitly

$$\pi_{L,N}(\eta) = \begin{cases} 0 & , \eta \notin X_{L,N} \\ \frac{\rho^N (1-\rho)^{L-N}}{\binom{L}{N} \rho^N (1-\rho)^{L-N}} = 1/\binom{L}{N} & , \eta \in X_{L,N} \end{cases}, \quad (2.41)$$

and  $\pi_{L,N}$  is uniform on  $X_{L,N}$  and in particular independent of  $\rho$ . We can write the grand-canonical product measures  $\nu_\rho$  as convex combinations

$$\nu_\rho = \sum_{N=0}^L \binom{L}{N} \rho^N (1-\rho)^{L-N} \pi_{L,N}, \quad (2.42)$$

but this is not possible for the  $\pi_{L,N}$  since they concentrate on irreducible subsets  $X_{L,N} \subsetneq X_L$ . Thus for the ASEP on  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  we have

$$\mathcal{I}_e = \{\pi_{L,N} : N = 0, \dots, L\} \quad (2.43)$$

given by the canonical measures. So for each value of the conserved quantity  $\Sigma_L$  we have an extremal stationary measure and these are the only elements of  $\mathcal{I}_e$ . The latter follows from

$$X_L = \bigcup_{N=0}^L X_{L,N} \quad \text{and} \quad \text{irreducibility on each } X_{L,N}. \quad (2.44)$$

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<sup>1</sup>In fact, absolute continuity and existence of a density are equivalent by the Radon-Nikodym theorem (see e.g. [13] Thm. 2.10).

In fact, suppose that for some  $\lambda \in (0, 1)$  and  $\mu_1, \mu_2 \in \mathcal{I}$

$$\pi_{L,N} = \lambda\mu_1 + (1 - \lambda)\mu_2 . \quad (2.45)$$

Then for all measurable  $Y \subseteq X$  with  $Y \cap X_{L,N} = \emptyset$  we have

$$0 = \pi_{L,N}(Y) = \lambda\mu_1(Y) + (1 - \lambda)\mu_2(Y) , \quad (2.46)$$

which implies that  $\mu_1(Y) = \mu_2(Y) = 0$ . So  $\mu_1, \mu_2 \in \mathcal{I}$  concentrate on  $X_{L,N}$  and thus  $\mu_1 = \mu_2 = \pi_{L,N}$  by uniqueness of  $\pi_{L,N}$  on  $X_{L,N}$ . So the conservation law provides a decomposition of the state space  $X_L$  into irreducible non-communicating subsets.

In general, taking into account all symmetries and conservation laws provides a full decomposition of the state space, and on each part concentrates a unique extremal stationary measure. This is the appropriate notion of *uniqueness of stationary measures* (cf. Def. 1.11) for systems with conserved quantities/symmetries. In general, a symmetry  $T$  is said to be *broken*, if there exists  $\mu \in \mathcal{I}_e$  such that

$$\mu T(X) \in (0, \infty) \quad \text{and} \quad \frac{1}{\mu T(X)} \mu T \neq \mu . \quad (2.47)$$

This is usually a result of non-commuting symmetries. For instance for the ASEP on  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  the  $\pi_{L,N}$  are invariant under translations, but not under *CP-symmetry*, since CP-invariance and particle conservation do not commute (see problem sheet). CP-invariance is a simple symmetry and corresponds to particle-hole and space inversion, given by

$$\tau\eta(x) = 1 - \eta(L + 1 - x) . \quad (2.48)$$

A similar situation holds for the blocking measures for the ASEP on the infinite lattice  $\Lambda = \mathbb{Z}$ , which are not invariant under translations. Symmetry breaking is a form of non-uniqueness of stationary measures and is therefore often regarded as a *phase transition* in analogy to the theory of Gibbs measures. However the use of this analogy is doubtful, because if we take it literally phase transitions are all over the place (e.g. CP-invariance is broken even on finite lattices) and the concept becomes less and less useful.

### 2.3 Currents and conservation laws

Consider the one-dimensional ASEP on  $\Lambda = \mathbb{Z}$  or  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ . Remember the forward equation from Theorem 1.6

$$\frac{d}{dt} S(t)f = S(t)\mathcal{L}f \quad \text{which holds for all } f \in C_0(X) . \quad (2.49)$$

Integrating w.r.t. the initial distribution  $\mu$  the equation becomes

$$\frac{d}{dt} \mu(S(t)f) = \mu(S(t)\mathcal{L}f) = (\mu S(t))(\mathcal{L}f) . \quad (2.50)$$

Using  $f(\eta) = \eta(x)$  and writing  $\mu_t := \mu S(t)$  for the distribution at time  $t$  we have

$$\mu_t(f) = \mathbb{E}^\mu(\eta_t(x)) =: \rho(x, t) \quad (2.51)$$

for the *particle density* at site  $x$  at time  $t$ . Note that  $\eta(x)$  is a cylinder function and we have

$$\begin{aligned} (\mathcal{L}f)(\eta) &= \sum_{y \in \Lambda} \left( p\eta(y)(1 - \eta(y+1)) + q\eta(y+1)(1 - \eta(y)) \right) (f(\eta^{y,y+1}) - f(\eta)) = \\ &= -p\eta(x)(1 - \eta(x+1)) + q\eta(x+1)(1 - \eta(x)) \\ &\quad -q\eta(x)(1 - \eta(x-1)) + p\eta(x-1)(1 - \eta(x)). \end{aligned} \quad (2.52)$$

Taking expectations w.r.t.  $\mu_t$  and writing

$$\mu_t(\eta(x)(1 - \eta(x+1))) = \mu_t(1_x 0_{x+1}) \quad (2.53)$$

we get with (2.49)

$$\frac{d}{dt} \rho(x, t) = \underbrace{p\mu_t(1_{x-1} 0_x) + q\mu_t(0_x 1_{x+1})}_{\text{gain}} - \underbrace{p\mu_t(1_x 0_{x+1}) + q\mu_t(0_{x-1} 1_x)}_{\text{loss}}. \quad (2.54)$$

**Definition 2.3** The *average current* of particles across a directed edge  $(x, y)$  on a general lattice (graph) is given by

$$j(x, y, t) := \mu_t(c(x, y, \eta) - c(y, x, \eta)). \quad (2.55)$$

For the ASEP this is non-zero only across nearest-neighbour bonds and given by

$$j(x, x+1, t) = p\mu_t(1_x 0_{x+1}) - q\mu_t(0_x 1_{x+1}). \quad (2.56)$$

Then we can write, using the lattice derivative  $\nabla_x j(x-1, x, t) = j(x, x+1, t) - j(x-1, x, t)$ ,

$$\frac{d}{dt} \rho(x, t) + \nabla_x j(x-1, x, t) = 0 \quad (2.57)$$

which is the (lattice) *continuity equation*. It describes the time evolution of the density  $\rho(x, t)$  in terms of higher order (two-point) correlation functions. The form of this equation implies that the particle density is conserved, i.e. on the finite lattice  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  with periodic boundary conditions we have

$$\frac{d}{dt} \sum_{x \in \Lambda_L} \rho(x, t) = - \sum_{x \in \Lambda_L} \nabla_x j(x-1, x, t) = 0. \quad (2.58)$$

In general on any finite subset  $A \in \Lambda$

$$\frac{d}{dt} \sum_{x \in A} \rho(x, t) = - \sum_{x \in \partial A} \nabla_x j(x-1, x, t), \quad (2.59)$$

where  $\partial A$  is the boundary of  $A$ . The other terms in the telescoping sum on the right-hand side cancel, which is a primitive version of Gauss' integration theorem (we have not been very careful with the notation at the boundary here).

In the special case  $p = q$  (2.57) simplifies significantly. Let's take  $p = q = 1$ , then adding and subtracting an auxiliary term we see

$$\begin{aligned} j(x, x+1, t) &= \mu_t(1_x 0_{x+1}) + \mu_t(1_x 1_{x+1}) - \mu_t(1_x 1_{x+1}) - \mu_t(0_x 1_{x+1}) = \\ &= \mu_t(1_x) - \mu_t(1_{x+1}) = \rho(x, t) - \rho(x+1, t) = -\nabla_x \rho(x, t). \end{aligned} \quad (2.60)$$

So the current is given by the lattice derivative of the density, and (2.57) turns into a closed equation

$$\frac{d}{dt}\rho(x, t) = \Delta_x \rho(x, t) = \rho(x-1, t) - 2\rho(x, t) + \rho(x+1, t). \quad (2.61)$$

Thus the particle density of the SSEP behaves like the probability density of a single simple random walk with jump rates  $p = q = 1$ .

To describe this behaviour on large scales we scale the lattice constant by a factor of  $1/L$  and embed it in the continuum, i.e.  $\frac{1}{L}\Lambda \subseteq \mathbb{R}$  and  $\frac{1}{L}\Lambda_L \subseteq \mathbb{T} = \mathbb{R}/\mathbb{Z}$  for the torus. Using the macroscopic space variable  $y = x/L \in \mathbb{R}, \mathbb{T}$  we define

$$\tilde{\rho}(y, t) := \rho([yL], t) \quad (2.62)$$

for the macroscopic *density field* and use a Taylor expansion

$$\rho(x \pm 1, t) = \tilde{\rho}(y \pm \frac{1}{L}, t) = \tilde{\rho}(y, t) \pm \frac{1}{L}\partial_y \tilde{\rho}(y, t) + \frac{1}{2L^2}\partial_y^2 \tilde{\rho}(y, t) + o(\frac{1}{L^2}) \quad (2.63)$$

to compute the lattice Laplacian in (2.61). This leads to

$$\Delta_x \rho(x, t) = \frac{1}{L^2}\partial_y^2 \tilde{\rho}(y, t), \quad (2.64)$$

since first order terms vanish due to symmetry. In order to get a non-degenerate equation in the limit  $L \rightarrow \infty$ , we have to scale time as  $s = t/L^2$ . This corresponds to speeding up the process by a factor of  $L^2$ , in order to see diffusive motion of the particles on the scaled lattice. Using both in (2.61) we obtain in the limit  $L \rightarrow \infty$

$$\partial_s \tilde{\rho}(y, s) = \partial_y^2 \tilde{\rho}(y, s), \quad (2.65)$$

the *heat equation*, describing the diffusion of particles on large scales.

If we use a stationary measure  $\mu_t = \mu$  in the continuity equation (2.57) we get

$$0 = \frac{d}{dt}\mu(1_x) = j(x-1, x) - j(x, x+1), \quad (2.66)$$

which implies that the *stationary current*  $j(x, x+1) := p\mu(1_x 0_{x+1}) - q\mu(0_x 1_{x+1})$  is site-independent. Since we know the stationary measures for the ASEP from the previous section we can compute it explicitly. For the homogeneous product measure  $\mu = \nu_\rho$  we get

$$j(x, x+1) := p\nu_\rho(1_x 0_{x+1}) - q\nu_\rho(0_x 1_{x+1}) = (p-q)\rho(1-\rho) =: \phi(\rho), \quad (2.67)$$

which is actually just a function of the total particle density  $\rho \in [0, 1]$ . We can use this to arrive at a scaling limit of the continuity equation for the asymmetric case  $p \neq q$ . We use the same space scaling  $y = x/L$  as above and write

$$\nabla_x j(x-1, x, t) = \frac{1}{L}\partial_y \tilde{j}(y - \frac{1}{L}, y, t) + o(\frac{1}{L}), \quad (2.68)$$

with a similar notation  $\tilde{j}$  as for  $\tilde{\rho}$  above. In the asymmetric case the first order terms in the spatial derivative do not vanish and we have to scale time as  $s = t/L$ , speeding up the process only by a factor  $L$  to see ballistic motion. In the limit  $L \rightarrow \infty$  this leads to the *conservation law (PDE)*

$$\partial_s \tilde{\rho}(y, s) + \partial_y \tilde{j}(y, s) = 0, \quad (2.69)$$

where we have redefined  $\tilde{j}$  as

$$\tilde{j}(y, s) := \lim_{L \rightarrow \infty} j([yL] - 1, [yL], sL). \quad (2.70)$$

Since we effectively take microscopic time  $t = sL \rightarrow \infty$  in that definition, it is plausible to assume that

$$\tilde{j}(y, s) = \phi(\tilde{\rho}(y, s)) \quad (2.71)$$

is in fact the stationary current corresponding to the local density  $\tilde{\rho}(y, s)$ . This is equivalent to the process becoming locally stationary in the limit  $L \rightarrow \infty$ , the only (slowly) varying quantity remaining on a large scale is the macroscopic density field. Local stationarity (also called local equilibrium) implies for example

$$\mu S(sL)(1_{[yL]}0_{[yL]+1}) \rightarrow \nu_{\tilde{\rho}(y,s)}(1_00_1) = \tilde{\rho}(y, s)(1 - \tilde{\rho}(y, s)) \quad \text{as } L \rightarrow \infty. \quad (2.72)$$

**Definition 2.4** The ASEP on  $\frac{1}{L}\mathbb{Z}$  or  $\frac{1}{L}\mathbb{Z}/L\mathbb{Z}$  with initial distribution  $\mu$ , such that

$$\tilde{\rho}(y, 0) = \lim_{L \rightarrow \infty} \mu(1_{[yL]}) \quad (2.73)$$

exists, is in *local equilibrium* if

$$\mu S(Ls)\tau_{-[yL]} \rightarrow \nu_{\tilde{\rho}(y,s)} \quad \text{weakly (locally), as } L \rightarrow \infty, \quad (2.74)$$

where  $\tilde{\rho}(y, s)$  is a solution of the *Burgers* equation

$$\partial_s \tilde{\rho}(y, s) + \partial_y \phi(\tilde{\rho}(y, s)) = 0 \quad \text{where } \phi(\rho) = (p - q)\rho(1 - \rho), \quad (2.75)$$

with initial condition  $\tilde{\rho}(y, 0)$ .

By local weak convergence we mean

$$\mu S(Ls)\tau_{-[yL]}(f) \rightarrow \nu_{\tilde{\rho}(y,s)}(f) \quad \text{for all } f \in C_0(X). \quad (2.76)$$

Local equilibrium has been established rigorously for the ASEP in a so-called *hydrodynamic limit*, the formulation of this result requires the following definition.

**Definition 2.5** For each  $t \geq 0$  we define the *empirical measure*

$$\pi_t^L := \frac{1}{L} \sum_{x \in \Lambda} \eta_t(x) \delta_{x/L} \in \mathcal{M}(\mathbb{R}) \text{ or } \mathcal{M}(\mathbb{T}), \quad (2.77)$$

and the measure-valued process  $(\pi_t^L : t \geq 0)$  is called the *empirical process*.

The  $\pi_t^L$  describe the discrete particle densities on  $\mathbb{R}, \mathbb{T}$ . They are (random) measures depending on the configurations  $\eta_t$  and for  $A \subseteq \mathbb{R}, \mathbb{T}$  we have

$$\pi_t^L(A) = \frac{1}{L} \left( \# \text{ of particles in } A \cap \frac{1}{L}\Lambda \text{ at time } t \right). \quad (2.78)$$

**Theorem 2.7** Consider the ASEP  $(\eta_t : t \geq 0)$  on the lattice  $\frac{1}{L}\mathbb{Z}$  or  $\frac{1}{L}\mathbb{Z}/L\mathbb{Z}$  with initial distribution  $\mu$  which has a limiting density  $\tilde{\rho}(y, 0)$  analogous to (2.73). Then as  $L \rightarrow \infty$

$$\pi_{sL}^L \rightarrow \tilde{\rho}(\cdot, s) dy \quad \text{weakly, in probability,} \quad (2.79)$$

where  $\tilde{\rho}(y, s)$  is a solution of (2.75) on  $\mathbb{R}$  or  $\mathbb{T}$  with initial condition  $\tilde{\rho}(y, 0)$ .

Here weak convergence means that for every  $g \in C_0(\mathbb{R})$  continuous with compact support

$$\pi_{sL}^L(g) = \frac{1}{L} \sum_{x \in \Lambda} g(x/L) \eta_t(x) \rightarrow \int_{\mathbb{R}, \mathbb{T}} g(y) \tilde{\rho}(y, s) dy . \quad (2.80)$$

The left-hand side is still random, and convergence holds in probability, i.e. for all  $\epsilon > 0$

$$\mathbb{P}^\mu \left( \left| \frac{1}{L} \sum_{x \in \Lambda} g(x/L) \eta_t(x) - \int_{\mathbb{R}, \mathbb{T}} g(y) \tilde{\rho}(y, s) dy \right| > \epsilon \right) \rightarrow 0 \quad \text{as } L \rightarrow \infty . \quad (2.81)$$

The proof is far beyond the scope of this course. The basic idea consists of two steps:

- For large  $L$  the empirical distribution  $\pi_{sL}$  should be close to the distribution  $\mu S(sL)$  at time  $sL$  due to a law of large numbers effect resulting from the space scaling.
- Establish a local equilibrium according to Def. 2.4, which should follow from the time scaling and the process reaching local stationarity.

Of course space and time scaling are carried out simultaneously. Both approximations above will give error terms depending on  $L$ , which have to be shown to vanish in the limit  $L \rightarrow \infty$ . Hydrodynamic limits are still an area of major research and technically quite involved. Relevant results and references can be found in [11] Chapter 8. The above result was first proved in [21] for the TASEP ( $q = 0$ ), and in [22] for a more general class of models using attractivity, a concept that will be discussed in Section 4.

## 2.4 Hydrodynamics and the dynamic phase transition

In the previous section we were often talking about solutions to the Burgers equation (2.75), not mentioning that it is far from clear whether that equation actually has a unique solution. A useful method to solve a *hyperbolic conservation law* of the form

$$\partial_t \rho(x, t) + \partial_x \phi(\rho(x, t)) = 0 , \quad \rho(x, 0) = \rho_0(x) \quad (2.82)$$

with general *flux function*  $\phi$  are characteristics (see [23] for full details). In this section we write again  $\rho$  for the macroscopic density to avoid notational overload, the notation  $\tilde{\rho}$  was only introduced to make the scaling argument clear in the previous section. We consider (2.82) for  $x \in \mathbb{R}$  or with periodic boundary conditions  $x \in \mathbb{T}$ .

**Definition 2.6** A curve  $x : [0, \infty) \rightarrow \mathbb{R}, \mathbb{T}$  with  $t \mapsto x(t)$  is a *characteristic* for the PDE (2.82) if

$$\frac{d}{dt} \rho(x(t), t) = 0 \quad \text{for all } t \geq 0 , \quad (2.83)$$

i.e. the solution is constant along  $x(t)$  and given by the initial conditions,  $\rho(x(t), t) = \rho_0(x(0))$ .

Using the PDE (2.82) to compute the total derivative we get

$$\begin{aligned} \frac{d}{dt} \rho(x(t), t) &= \partial_t \rho(x(t), t) + \partial_x \rho(x(t), t) \dot{x}(t) = \\ &= -\phi'(\rho(x(t), t)) \partial_x \rho(x(t), t) + \partial_x \rho(x(t), t) \dot{x}(t) = 0 , \end{aligned} \quad (2.84)$$

which implies that

$$\dot{x}(t) = \phi'(\rho(x(t), t)) = \phi'(\rho_0(x(0))) \quad (2.85)$$

is a constant given by the derivative of the flux function. This is called the *characteristic velocity*  $u(\rho)$ , and for the ASEP we have

$$u(\rho) = \phi'(\rho) = (p - q)(1 - 2\rho). \quad (2.86)$$

It turns out (see [23]) that a general solution theory for hyperbolic conservation laws of the form (2.82) can be based on understanding the solutions to the *Riemann problem*, which is given by step initial data

$$\rho_0(x) = \begin{cases} \rho_l & , x \leq 0 \\ \rho_r & , x > 0 \end{cases}. \quad (2.87)$$

Discontinuous solutions of a PDE have to be understood in a weak sense.

**Definition 2.7**  $\rho : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$  is a *weak solution* to the conservation law (2.82) if  $\rho \in L^1_{loc}(\mathbb{R} \times [0, \infty))$  and for all  $\psi \in C^1(\mathbb{R} \times [0, \infty))$  with compact support and  $\psi(x, 0) = 0$ ,

$$\int_{\mathbb{R}} \int_0^\infty \partial_t \psi(x, t) \rho(x, t) dx dt + \int_{\mathbb{R}} \int_0^\infty f(\rho(x, t)) \partial_x \psi(x, t) dx dt = 0. \quad (2.88)$$

$L^1_{loc}$  means that for all compact  $A \subseteq \mathbb{R} \times [0, \infty)$ ,  $\int_A |\rho(x, t)| dx dt < \infty$ .

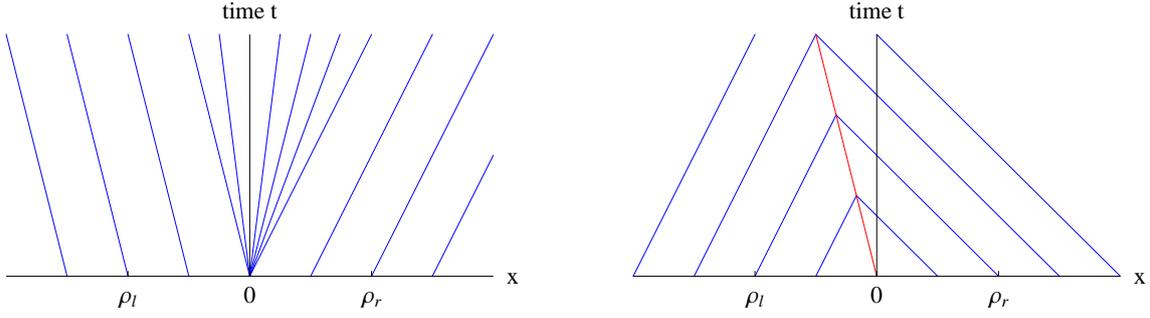
The characteristics do not necessarily uniquely determine a solution everywhere, so weak solutions are in general not unique. They can be undetermined or over-determined, and both cases appear already for the simple Riemann problem (2.87) (cf. Fig. 3). However, for a given initial density profile, the corresponding IPS which lead to the derivation of the PDE shows a unique time evolution on the macroscopic scale. This unique *admissible solution* can be recovered from the variety of weak solutions to (2.82) by several regularization methods. The *viscosity method* is directly related to the derivation of the continuum equation in a scaling limit. For every  $\epsilon > 0$  consider the equation

$$\partial_t \rho^\epsilon(x, t) + \partial_x \phi(\rho^\epsilon(x, t)) = \epsilon \partial_x^2 \phi(\rho^\epsilon(x, t)), \quad \rho^\epsilon(x, 0) = \rho_0(x). \quad (2.89)$$

This is a parabolic equation and has a unique smooth global solution for all  $t > 0$ , even when starting from non-smooth initial data  $\rho_0$ . This is due to the regularizing effect of the diffusive term (consider e.g. the heat equation starting with initial condition  $\delta_0(x)$ ), which captures the fluctuations in large finite IPS. The term can be interpreted as a higher order term of order  $1/L^2$  in the expansion (2.68), which disappears in the scaling limit from a particle system. Then one can define the unique admissible weak solution to (2.82) as

$$\rho(\cdot, t) := \lim_{\epsilon \rightarrow 0} \rho^\epsilon(\cdot, t) \quad \text{in } L^1_{loc}\text{-sense as above for all } t > 0. \quad (2.90)$$

It can be shown that this limit exists, and further that for one-dimensional conservation laws the precise form of the viscosity is not essential, i.e. one could also add the simpler term  $\epsilon \partial_x^2 \rho^\epsilon(x, t)$  leading to the same weak limit solution [23]. There are also other admissibility criteria for hyperbolic conservation laws such as entropy conditions, which can be shown to be equivalent to the viscosity method in one dimension. We do not discuss this further here, for details see [23].



**Figure 3:** Characteristics for the Riemann problem with  $\rho_l < \rho_r$  (left) showing a rarefaction fan, and  $\rho_l > \rho_r$  (right), showing a shock. The curve is shock location is shown in red and the speed is given by (2.93).

For the Riemann problem with flux function  $\phi(\rho) = (p - q)\rho(1 - \rho)$  for the ASEP, there are two basic scenarios for the time evolution of step initial data shown in Fig. 3. For  $\rho_r < \rho_l$  the characteristic speeds are  $u(\rho_r) > u(\rho_l)$ , and the characteristics point away from each other and open a cone of points  $(x, t)$  where the solution is not determined. The admissibility criteria described above show that the consistent solution in this case is given by the *rarefaction fan*

$$\rho(x, t) = \begin{cases} \rho_l & , x \leq u(\rho_l)t \\ \rho_r & , x > u(\rho_r)t \\ \rho_l + (x - tu(\rho_l)) \frac{\rho_l - \rho_r}{t(u(\rho_l) - u(\rho_r))} & , u(\rho_l)t < x \leq u(\rho_r)t \end{cases} . \quad (2.91)$$

So the step dissolves and the solution interpolates linearly between the points uniquely determined by the characteristics. An illustrative extreme version of this case is the 'traffic light problem', where  $\rho_l = 1$  and  $\rho_r = 0$  corresponding to cars piling up behind a red traffic light. When the traffic light turns green not all cars start moving at once, but the density gradually decreases following a continuous linear profile like in real situations.

For  $\rho_r > \rho_l$  we have  $u(\rho_r) < u(\rho_l)$  and the characteristics point towards each other so that the solution is over-determined in a cone around the origin. Admissibility criteria show that in this case the step is stable, called a *shock solution*,

$$\rho(x, t) = \begin{cases} \rho_l & , x \leq vt \\ \rho_r & , x > vt \end{cases} . \quad (2.92)$$

In the traffic analogy shocks correspond sharp ends of traffic jams, where density and flow change rather abruptly. The *shock speed*  $v = v(\rho_l, \rho_r)$  can be derived by the conservation of mass. The average number of particles  $m \geq 0$  transported through the shock in negative direction during a time interval  $\Delta t$  is given by  $m = \Delta t(\phi(\rho_r) - \phi(\rho_l))$ . If  $m > 0$  ( $m < 0$ ) this causes the shock to move with positive (negative) speed  $v$ . Therefore  $m$  is also given by  $m = \Delta t v (\rho_r - \rho_l)$  leading to

$$v(\rho_l, \rho_r) = \frac{\phi(\rho_r) - \phi(\rho_l)}{\rho_r - \rho_l} . \quad (2.93)$$

As mentioned before, understanding the Riemann problem is sufficient to construct solutions to general initial data by approximations with piecewise constant functions.

In the following we will use our knowledge on solutions to the Riemann problem to understand the time evolution of the ASEP with step initial distribution

$$\mu = \nu_{\rho_l, \rho_r} \text{ product measure with } \nu_{\rho_l, \rho_r}(\eta(x)) = \begin{cases} \rho_l & , x \leq 0 \\ \rho_r & , x > 0 \end{cases} . \quad (2.94)$$

**Theorem 2.8** For the ASEP on  $\Lambda = \mathbb{Z}$  with  $p > q$  we have as  $t \rightarrow \infty$

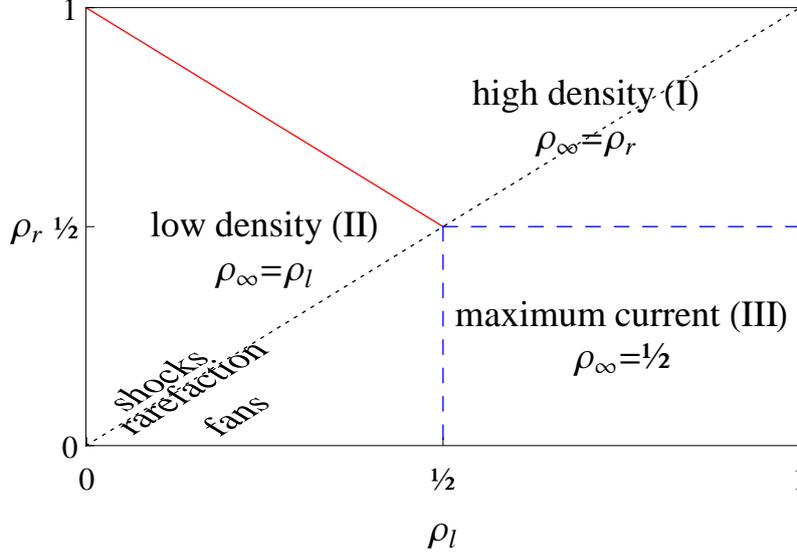
$$\nu_{\rho_l, \rho_r} S(t) \rightarrow \begin{cases} \nu_{\rho_r} & , \rho_r \geq \frac{1}{2}, \rho_l > 1 - \rho_r & \text{(I)} \\ \nu_{\rho_l} & , \rho_l \leq \frac{1}{2}, \rho_r < 1 - \rho_r & \text{(II)} \\ \nu_{1/2} & , \rho_l \geq \frac{1}{2}, \rho_r \leq \frac{1}{2} & \text{(III)} \end{cases} \quad (2.95)$$

**Proof.** by studying shock and rarefaction fan solutions of the conservation law (2.82).

Note that all the limiting distributions are stationary product measures of the ASEP, as required by Theorem 1.9. But depending on the initial distribution, the system selects different stationary measures in the limit  $t \rightarrow \infty$ , which do not depend smoothly on  $\rho_l$  and  $\rho_r$ . Therefore this phenomenon is called a *dynamic phase transition*. The set  $\mathcal{I}$  of stationary measures is not changed, but the long-time behaviour of the process depends on the initial conditions in a non-smooth way. This behaviour can be captured in a *phase diagram*, whose axes are given by the (fixed) parameters of our problem,  $\rho_l$  and  $\rho_r$ . We choose the limiting density

$$\rho_\infty := \lim_{t \rightarrow \infty} \nu_{\rho_l, \rho_r} S(t)(\eta(0)) \quad (2.96)$$

as the *order parameter*, which characterizes the phase transition. The different *phase regions* correspond to areas of qualitatively distinct behaviour of  $\rho_\infty$  as a function of  $\rho_l$  and  $\rho_r$ .



Above the dashed diagonal the solutions of the conservation law (2.82) are given by shocks, and below by rarefaction fans. Analysing the different cases reveals the following behaviour:

- (I) *High density phase:* The limiting density  $\rho_\infty = \rho_r \geq 1/2$ , since particles drifting to the right are jamming behind the region of high density.

- (II) *Low density phase*: The limiting density is  $\rho_\infty = \rho_l \leq 1/2$ , since particles can drift to the right without jamming.
- (III) *Maximum current phase*: The solution to the PDE is a rarefaction fan with negative (positive) characteristic velocity  $u$  on the left (right). Thus the limiting density is given by the density  $1/2$  with vanishing  $u(1/2) = 0$ .

The dashed blue line is a continuous phase transition line, i.e. crossing this line the order parameter  $\rho_\infty(\rho_l, \rho_r)$  is continuous. The full red line is a first order transition line, across which the order parameter jumps from  $\rho_l < 1/2$  to  $\rho_r > 1/2$ . The exact behaviour of the system on that line is given by

$$\nu_{\rho_l, \rho_r} S(t) \rightarrow \frac{1}{2} \nu_{\rho_l} + \frac{1}{2} \nu_{\rho_r} . \quad (2.97)$$

So the limiting distribution is a mixture, and with equal probability all local observables are determined by the left or the right product measure. Formally this leads to  $\rho_\infty = 1/2$  as  $\rho_l + \rho_r = 1$ , but this is misleading. The local density at the origin averaged over space is typically either  $\rho_l$  or  $\rho_r$  with equal probability, but never  $1/2$  as it would be for  $\nu_{1/2}$ . This difference can be detected by looking at a higher order correlation functions such as  $\eta(0)\eta(1)$ , which leads to

$$\left(\frac{1}{2}\nu_{\rho_l} + \frac{1}{2}\nu_{\rho_r}\right)(\eta(0)\eta(1)) = \frac{1}{2}(\rho_l^2 + \rho_r^2) , \quad (2.98)$$

as opposed to  $\nu_{1/2}(\eta(0)\eta(1)) = 1/4$ . More details on this in the context of a similar phase transition are derived and discussed in the next subsection for a finite system with open boundary conditions.

The characteristics of the hyperbolic conservation law (2.82) provide a powerful tool to describe the transport properties of an IPS on a macroscopic scale. Their counterpart on a microscopic lattice scale are so-called *second class particles*, which move randomly along the characteristics depending on the local density. Since characteristics meet in shocks, second class particles are attracted by shocks, and provide a good microscopic marker for the position of a shock. This is important since a priori shocks do not look sharp on the lattice scale do not have a well defined location. Therefore second class particles are an important concept and have been studied in great detail (see e.g. [5] Section III.2 and references therein).

## 2.5 \*Open boundaries and matrix product ansatz

In the following we consider the ASEP on the lattice  $\Lambda_L = \{1, \dots, L\}$  with *open boundary conditions*. So in addition to the bulk rates

$$10 \xrightarrow{p} 01 \quad \text{and} \quad 01 \xrightarrow{q} 10 , \quad (2.99)$$

we have to specify boundary rates for creation and annihilation of particles at sites  $x = 1$  and  $L$ ,

$$|0 \xrightarrow{\alpha} |1 , \quad |1 \xrightarrow{\gamma} |0 , \quad |1 \xrightarrow{\beta} 0| \quad \text{and} \quad 0| \xrightarrow{\delta} 1| . \quad (2.100)$$

In principle we are free to choose  $\alpha, \beta, \gamma$  and  $\delta \geq 0$  independently. We would like to model the situation where the system is coupled to *particle reservoirs* at both ends with densities  $\rho_l$  and  $\rho_r \in [0, 1]$ , which implies

$$\alpha = \rho_l p , \quad \gamma = q(1 - \rho_l) , \quad \beta = p(1 - \rho_r) \quad \text{and} \quad \delta = q\rho_r . \quad (2.101)$$

The generator of the process is then given by the sum

$$\begin{aligned}
\mathcal{L}f(\eta) &= \mathcal{L}_{bulk}f(\eta) + \mathcal{L}_{bound}f(\eta) = \\
&= \sum_{x=1}^{L-1} \left( p\eta(x)(1 - \eta(x+1)) - q\eta(x+1)(1 - \eta(x)) \right) (f(\eta^{x,x+1}) - f(\eta)) + \\
&\quad + \left( p\rho_l(1 - \eta(1)) - q\eta(1)(1 - \rho_l) \right) (f(\eta^1) - f(\eta)) + \\
&\quad + \left( p\eta(L)(1 - \rho_r) - q\rho_r(1 - \eta(L)) \right) (f(\eta^L) - f(\eta)) .
\end{aligned} \tag{2.102}$$

Note that for  $\rho_l, \rho_r \in (0, 1)$  particles are created and destroyed at the boundaries, and the number of particles is not conserved. The ASEP on  $\Lambda_L$  is thus a finite state irreducible Markov chain on  $X_L = \{0, 1\}^{\Lambda_L}$ . Therefore with Prop. 1.10 the process is ergodic and has a unique stationary measure  $\mu_L = \mu_L(\rho_l, \rho_r)$  depending on the boundary parameters.

Following the analysis of the previous section, the scaled stationary density profile

$$\rho(y) := \lim_{L \rightarrow \infty} \mu_L(1_{[yL]}) \quad \text{with } y \in [0, 1] \tag{2.103}$$

should be a stationary solution of the conservation law (2.82). This is given by the boundary value problem

$$0 = \partial_y \phi(\rho(y)) = (p - q)(1 - 2\rho(y))\partial_y \rho(y) \quad \text{with } \rho(0) = \rho_l, \rho(1) = \rho_r, \tag{2.104}$$

which has constant solutions. This is a first order equation which is not well posed having two boundary conditions  $\rho_l \neq \rho_r$ . So jumps at the boundary cannot be avoided and obviously the solution can be any arbitrary constant. Again one can apply the viscosity method as in the previous section to get a unique solution for all  $\epsilon > 0$ . Adding a second order term to (2.104) yields a well posed parabolic equation with a unique solution  $\rho^\epsilon(y)$ , from which we retrieve the admissible stationary profile  $\rho(y)$  in the limit  $\epsilon \rightarrow 0$ .

Understanding the motion of shocks and rarefaction fans, we can derive the stationary profile  $\rho(y)$  also from the time dependent solution  $\rho(y, t)$  in the limit  $t \rightarrow \infty$ . As initial condition we can choose

$$\rho_0(y) = \begin{cases} \rho_l & , 0 \leq y \leq a \\ \rho_r & , a < y \leq 1 \end{cases} \quad \text{for some } a \in (0, 1) . \tag{2.105}$$

Then the macroscopic stationary profile  $\rho(y)$  is given by a constant  $\rho_{bulk}$  that corresponds exactly to the densities  $\rho_\infty$  observed in Theorem 2.8 for the infinite system, i.e.

$$\rho_{bulk} = \begin{cases} \rho_r & , \rho_r \geq \frac{1}{2}, \rho_l > 1 - \rho_r & \text{(high density)} \\ \rho_l & , \rho_l \leq \frac{1}{2}, \rho_r < 1 - \rho_r & \text{(low density)} \\ 1/2 & , \rho_l \geq \frac{1}{2}, \rho_r \leq \frac{1}{2} & \text{(maximum current)} \end{cases} . \tag{2.106}$$

In contrast to the previous section this is only correct in the scaling limit. For finite  $L$  boundary effects produce visible deviations and in particular correlations. So the stationary measure is not of product form, except for the trivial case  $\rho_l = \rho_r$ .

A very powerful ansatz to represent the non-product stationary distribution in this case is given by using products of matrices.

**Theorem 2.9** Consider the ASEP on  $\Lambda_L = \{0, \dots, L\}$  with boundary densities  $\rho_l, \rho_r \in (0, 1)$  and bulk rates  $p, q$ . Suppose that the (possibly infinite) matrices  $D, E$  and vectors  $\mathbf{w}, \mathbf{v}$  satisfy

$$\begin{aligned} pDE - qED &= D + E \\ \mathbf{w}^T (\rho_l p E - (1 - \rho_l) q D) &= \mathbf{w} \\ ((1 - \rho_r) p D - \rho_r q E) \mathbf{v} &= \mathbf{v} . \end{aligned} \quad (2.107)$$

These relations are called a **quadratic algebra**. For  $\eta \in X_L$  put

$$g_L(\eta) = \mathbf{w}^T \prod_{x=1}^L (\eta(x) D + (1 - \eta(x)) E) \mathbf{v} . \quad (2.108)$$

If this is a well defined number in  $\mathbb{R}$  for all  $\eta \in X_L$  and the normalization

$$Z_L = \sum_{\eta \in X_L} g_L(\eta) \neq 0 , \quad (2.109)$$

then the stationary distribution of the ASEP is given by  $\mu_L(\eta) = g_L(\eta)/Z_L$ .

**Remark.** The matrices  $D, E$  and the vectors  $\mathbf{v}, \mathbf{w}$  are purely auxiliary and have no a priori interpretation in terms of the particle system. In a field theoretic interpretation of the process, the matrices can be related to creation and annihilation operators. For more details on this and the matrix product ansatz in general see [24] and references therein.

**Proof.**  $(\eta_t : t \geq 0)$  is a finite state irreducible MC and has a unique stationary measure  $\mu_L$ , given by the stationary solution of the master equation

$$\frac{d}{dt} \mu_L(\eta) = 0 = \sum_{\eta' \in X_L} (\pi_L(\eta') c(\eta', \eta) - \pi_L(\eta) c(\eta, \eta')) \quad \text{for all } \eta \in X_L . \quad (2.110)$$

(Recal that this is the stationarity condition  $\mu_L(\mathcal{L}f) = 0$  for  $f = \mathbb{1}_\eta$ .)

Therefore it suffices to show that  $g_L$  given in (2.108) fulfills the master equation, then it can automatically be normalized. In our case the (unnormalized) individual terms in the sum are of the form

$$g_L(\eta^{x,x+1}) c(x, x+1, \eta^{x,x+1}) - g_L(\eta) c(x, x+1, \eta) \quad (2.111)$$

for the bulk and similar for the boundaries. They can be simplified using the quadratic algebra (2.107). Using the first rule we get for the bulk

$$\begin{aligned} g_L(\dots, 0, 1, \dots) q - g_L(\dots, 1, 0, \dots) p &= -g_{L-1}(\dots, 1, \dots) - g_{L-1}(\dots, 0, \dots) \quad \text{and} \\ g_L(\dots, 1, 0, \dots) p - g_L(\dots, 0, 1, \dots) q &= g_{L-1}(\dots, 1, \dots) + g_{L-1}(\dots, 0, \dots) . \end{aligned} \quad (2.112)$$

In general we can write for  $x \in \{1, \dots, L-1\}$

$$\begin{aligned} g_L(\eta^{x,x+1}) c(\eta^{x,x+1}, \eta) - g_L(\eta) c(\eta, \eta^{x,x+1}) &= (1 - 2\eta(x)) g_{L-1}(\dots, \eta(x-1), \eta(x), \dots) - \\ &\quad - (1 - 2\eta(x+1)) g_{L-1}(\dots, \eta(x), \eta(x+1), \dots) . \end{aligned} \quad (2.113)$$

For the boundaries we get analogously

$$\begin{aligned} g_L(\eta^1) c(1, \eta^1) - g_L(\eta) c(1, \eta) &= -(1 - 2\eta(1)) g_{L-1}(\eta(2), \dots) \quad \text{and} \\ g_L(\eta^L) c(L, \eta^L) - g_L(\eta) c(L, \eta) &= (1 - 2\eta(L)) g_{L-1}(\dots, \eta(L-1)) . \end{aligned} \quad (2.114)$$

The sum over all  $x \in \Lambda_L$  corresponds to the right-hand side of (2.110), and vanishes since it is a telescoping sum.  $\square$

If the system is reversible then the terms (2.111) vanish individually. In the general non-reversible case they are therefore called *defects from reversibility*, and the quadratic algebra provides a simplification of those in terms of distributions for smaller system sizes.

In terms of the matrices, the normalization is given by

$$Z_L = \mathbf{w}^T C^L \mathbf{v} \quad \text{with} \quad C = D + E. \quad (2.115)$$

Correlation functions can be computed as

$$\rho(x) = \mu_L(1_x) = \frac{\mathbf{w}^T C^{x-1} D C^{L-x} \mathbf{v}}{\mathbf{w}^T C^L \mathbf{v}}, \quad (2.116)$$

or for higher order with  $x > y$ ,

$$\mu_L(1_x 1_y) = \frac{\mathbf{w}^T C^{x-1} D C^{y-x-1} D C^{L-y} \mathbf{v}}{\mathbf{w}^T C^L \mathbf{v}}. \quad (2.117)$$

In particular for the stationary current we get

$$j(x) = \frac{\mathbf{w}^T C^{x-1} (pDE - qED) C^{L-x-1} \mathbf{v}}{\mathbf{w}^T C^L \mathbf{v}} = \frac{\mathbf{w}^T C^{L-1} \mathbf{v}}{\mathbf{w}^T C^L \mathbf{v}} = \frac{Z_{L-1}}{Z_L}, \quad (2.118)$$

which is independent of the lattice site as expected from (2.66).

For  $\rho_l = \rho_r = \rho$  and  $p \neq q$  the algebra (2.107) is fulfilled by the one-dimensional matrices

$$E = \frac{1}{\rho(p-q)}, \quad D = \frac{1}{(1-\rho)(p-q)} \quad \text{and} \quad w = v = 1 \quad (2.119)$$

since

$$pDE - qED = \frac{(p-q)}{(p-q)^2 \rho(1-\rho)} = \frac{1}{(p-q)\rho(1-\rho)} = D + E = C \quad (2.120)$$

and  $\rho pE - (1-\rho)qD = (1-\rho)pD - \rho qE = 1$ .

$E, D \in \mathbb{R}$  implies that  $\mu_L$  is a product measure, and the density is hardly surprising,

$$\rho(x) = \rho(1) = \frac{DC^{L-1}}{C^L} = \rho \quad \text{so} \quad \mu_L = \nu_\rho. \quad (2.121)$$

In general  $\mu_L$  is a product measure if and only if there exist scalars  $E, D$  fulfilling the algebra (2.107), and it turns out that for  $\rho_l \neq \rho_r$  this is not the case. In general, there are several infinite representations possible, summarized in [24].

In the following let's focus on the totally asymmetric case  $p = 1, q = 0$  (TASEP) with  $\rho_l, \rho_r \in (0, 1)$ . The algebra simplifies to

$$DE = D + E, \quad \mathbf{w}^T E = \frac{1}{\rho_l} \mathbf{w}^T, \quad D\mathbf{v} = \frac{1}{1-\rho_r} \mathbf{v}, \quad (2.122)$$

and the question is what kind of matrices fulfill these relations.

**Proposition 2.10** *For  $p = 1, q = 0$ , if  $E, D$  are finite dimensional, then they commute.*

**Proof.** Suppose  $u$  satisfies  $Eu = u$ . Then by the first identity  $Du = Du + u$  and hence  $u = 0$ . Therefore  $E - I$  is invertible and we can solve the first identity

$$D = E(E - I)^{-1} \quad \text{which implies that } D \text{ and } E \text{ commute.} \quad \square \quad (2.123)$$

So to describe the non-product stationary measure  $\mu_L$ ,  $D$  and  $E$  have to be infinite dimensional. Possible choices are

$$D = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad (2.124)$$

with corresponding vectors

$$\mathbf{w}^T = \left( 1, \frac{1 - \rho_l}{\rho_l}, \left( \frac{1 - \rho_l}{\rho_l} \right)^2, \dots \right) \quad \text{and} \quad \mathbf{v}^T = \left( 1, \frac{\rho_r}{1 - \rho_r}, \left( \frac{\rho_r}{1 - \rho_r} \right)^2, \dots \right). \quad (2.125)$$

Correlation functions can be computed without using any representations by repeatedly applying the algebraic relations. Using the rules

$$\begin{aligned} DE &= C, \quad DC = D^2 + C, \quad CE = C + E^2 \quad \text{and} \\ \mathbf{w}^T E^k &= \frac{1}{\rho_l^k} \mathbf{w}^T, \quad D^k \mathbf{v} = \frac{1}{(1 - \rho_r)^k} \mathbf{v}, \end{aligned} \quad (2.126)$$

the probability of every configuration can be written as a combination of terms of the form  $Z_k = \mathbf{w}^T C^k \mathbf{v}$ . Explicit formulas can be derived which look rather complicated (see [24] and references therein), for the current we get the following limiting behaviour,

$$j = \frac{Z_{L-1}}{Z_L} \rightarrow \begin{cases} \rho_r(1 - \rho_r), & \rho_r > 1/2, \rho_l > 1 - \rho_r \\ \rho_l(1 - \rho_l), & \rho_l < 1/2, \rho_r < 1 - \rho_l \\ 1/4, & \rho_r \leq 1/2, \rho_l \geq 1/2 \end{cases} \quad \text{as } L \rightarrow \infty. \quad (2.127)$$

This is consistent with the hydrodynamic result. Using the matrix product ansatz, the following result can be shown rigorously.

**Theorem 2.11** *Suppose  $p = 1, q = 0$  and let  $x_L$  be a monotone sequence of integers such that  $x_L \rightarrow \infty$  and  $L - x_L \rightarrow \infty$  for  $L \rightarrow \infty$ . Then*

$$\mu_L \tau_{x_L} \rightarrow \begin{cases} \nu_{\rho_r}, & \rho_r > 1/2, \rho_l > 1 - \rho_r \\ \nu_{\rho_l}, & \rho_l < 1/2, \rho_r < 1 - \rho_l \\ \nu_{1/2}, & \rho_r \leq 1/2, \rho_l \geq 1/2 \end{cases} \quad \text{weakly, locally.} \quad (2.128)$$

If  $\rho_l < 1/2 < \rho_r$  and  $\rho_l + \rho_r = 1$  (first order transition line), then

$$\mu_L \tau_{x_L} \rightarrow (1 - a)\nu_{\rho_l} + a\nu_{\rho_r} \quad \text{where} \quad a = \lim_{L \rightarrow \infty} \frac{x_L}{L}. \quad (2.129)$$

**Proof.** see e.g. [5], Section III.3

Note that on the first order transition line, the result can be interpreted in terms of a shock measure with diffusing shock location, where the left part of the system has distribution  $\nu_{\rho_l}$  and the right part  $\nu_{\rho_r}$ . This phenomenon is called *phase coexistence*, and is described by a mixture of the form (2.129).

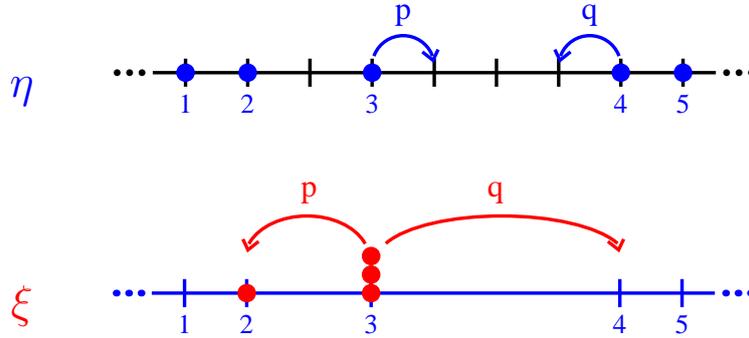
### 3 Zero-range processes

#### 3.1 From ASEP to ZRPs

Consider the ASEP on the lattice  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ . For each configuration  $\eta \in X_{L,N}$  with  $N = \sum_{x \in \Lambda_L} \eta(x)$  label the particles  $j = 1, \dots, N$  and let  $x_j \in \Lambda_L$  be the position of the  $j$ th particle. We attach the labels such that the positions are ordered  $x_1 < \dots < x_N$ . We map the configuration  $\eta$  to a configuration  $\xi \in \mathbb{N}^{\Lambda_N}$  on the lattice  $\Lambda_N = \{1, \dots, N\}$  by

$$\xi(j) = x_{j+1} - x_j - 1. \quad (3.1)$$

Here the lattice site  $j \in \Lambda_N$  corresponds to particle  $j$  in the ASEP and  $\xi_j \in \mathbb{N}$  to the distance to the next particle  $j + 1$ . Note that  $\eta$  and  $\xi$  are equivalent descriptions of an ASEP configuration up to the position  $x_1$  of the first particle.



As can be seen from the construction, the dynamics of the ASEP  $(\eta_t : t \geq 0)$  induce a process  $(\xi_t : t \geq 0)$  on the state space  $\mathbb{N}^{\Lambda_N}$  with rates

$$c(\xi, \xi^{j \rightarrow j+1}) = q(1 - \delta_{0, \xi(j)}) \quad \text{and} \quad c(\xi, \xi^{j \rightarrow j-1}) = p(1 - \delta_{0, \xi(j)}), \quad (3.2)$$

where we write  $\xi^{x \rightarrow y} = \begin{cases} \xi(x) - 1, & z = x \\ \xi(y) + 1, & z = y \\ \xi(z), & z \neq x, y \end{cases}$ .

Since the order of particles in the ASEP is conserved, we have  $\xi_t(j) \geq 0$  and therefore  $\xi_t \in \mathbb{N}^{\Lambda_N}$  for all  $t \geq 0$ . Note also that the number of  $\xi$ -particles is

$$\sum_{j \in \Lambda_N} \xi(j) = L - N = \text{number of holes in ASEP}, \quad (3.3)$$

which is conserved in time, and therefore  $(\xi_t : t \geq 0)$  is a lattice gas. There is no exclusion interaction for this process, i.e. the number of particles per site is not restricted. With analogy to quantum mechanics this process is sometimes called a *bosonic* lattice gas, whereas the ASEP is a *fermionic* system.

The  $\xi$ -process defined above is an example of a more general class of bosonic lattice gases, zero-range processes, which we introduce in the following. From now on we will switch back to our usual notation denoting configurations by  $\eta$  and lattice sizes by  $L$ .

**Definition 3.1** Consider a lattice  $\Lambda$  (any discrete set) and the state space  $X = \mathbb{N}^\Lambda$ . Let  $p(x, y)$  be the irreducible, finite range transition probabilities of a single random walker on  $\Lambda$  with  $p(x, x) = 0$ , called the *jump probabilities*. For each  $x \in \Lambda$  define the *jump rates*  $g_x : \mathbb{N} \rightarrow [0, \infty)$  as a non-negative function of the number of particles  $\eta(x)$  at site  $x$ , where

$$g_x(n) = 0 \quad \Leftrightarrow \quad n = 0 \quad \text{for all } x \in \Lambda. \quad (3.4)$$

Then the process  $(\eta_t : t \geq 0)$  on  $X$  defined by the generator

$$\mathcal{L}f(\eta) = \sum_{x, y \in \Lambda} g_x(\eta(x))p(x, y)(f(\eta^{x \rightarrow y}) - f(\eta)) \quad (3.5)$$

is called a *zero-range process (ZRP)*.

**Remarks.**

- ZRPs are interacting random walks with *zero-range* interaction, since the jump rate of a particle at site  $x \in \Lambda$  depends only on the number of particles  $\eta(x)$  at that site. The interpretation of the generator is that each site  $x$  loses a particle with rate  $g(\eta(x))$ , which then jumps to a site  $y$  with probability  $p(x, y)$ .
- The above  $\xi$ -process is a simple example of a ZRP with  $\Lambda = \mathbb{Z}/N\mathbb{Z}$  and

$$g_x(n) \equiv p + q, \quad p(x, x + 1) = \frac{q}{p + q} \quad \text{and} \quad p(x, x - 1) = \frac{p}{p + q}. \quad (3.6)$$

- On finite lattices  $\Lambda_L$  of size  $L$ , irreducibility of  $p(x, y)$  and (3.4) imply that ZRPs are irreducible finite state Markov chains on

$$X_{L, N} = \{\eta \in \mathbb{N}^{\Lambda_L} \mid \Sigma_L(\eta) = N\} \quad (3.7)$$

for all fixed particle numbers  $N \in \mathbb{N}$  (remember the shorthand  $\Sigma_L(\eta) = \sum_{x \in \Lambda_L} \eta(x)$ ). Therefore they have a unique stationary distribution  $\pi_{L, N}$  on  $X_{L, N}$ .

**Examples.**

- For the rates  $g_x(n) = g_x > 0$  for all  $n \geq 0$  and  $x \in \Lambda$  the ZRP can be interpreted as a network of  $M/M/1$  server queues<sup>1</sup>, where at each site  $x$  a single server completes jobs with rate  $g_x$  and passes them on to another server  $y$  according to  $p(x, y)$ .
- For the rates  $g_x(n) = g_x n$  for all  $x \in \Lambda$ , we have a network of  $M/M/\infty$  server queues, i.e. each queue can serve all the particles present at the same time. That means that each particle individually exits the queue at rate  $g_x$  independently of all others, leading to a total exit rate  $g_x n$ . (Remember from Section 1.1 that the sum of  $n$  independent  $PP(g_x)$  processes is a  $PP(g_x n)$  process.) Thus this corresponds to a system of independent random walkers moving with rates  $g_x p(x, y)$ .

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<sup>1</sup> $M/M/1$  means that a single server (1) receives input and generates output via continuous-time Markov processes (M), i.e. with exponential waiting time distributions. There are more general queuing systems with applications in traffic routing or process optimization (see e.g. [15] Chapter 11).

On infinite lattices the number of particles is in general also infinite, but as opposed to exclusion processes the local state space of a ZRP is  $\mathbb{N}$ . This is not compact, and therefore in general also  $X$  is not compact and the construction of the process with semigroups and generators given in Chapter 1 does not apply directly and has to be modified.

In addition to non-degeneracy (3.4) we assume a sub-linear growth of the jump rates, i.e.

$$\bar{g} := \sup_{x \in \Lambda} \sup_{n \in \mathbb{N}} |g_x(n+1) - g_x(n)| < \infty, \quad (3.8)$$

and restrict to the state space

$$X_\alpha = \{\eta \in \mathbb{N}^\Lambda \mid \|\eta\|_\alpha < \infty\} \quad \text{with} \quad \|\eta\|_\alpha = \sum_{x \in \Lambda} |\eta(x)| \alpha^{|x|} \quad (3.9)$$

for some  $\alpha \in (0, 1)$ . Let  $L(X) \subseteq C(X)$  be the set of *Lipshitz-continuous* test functions  $f : X_\alpha \rightarrow \mathbb{R}$ , i.e.

$$|f(\eta) - f(\zeta)| \leq l(f) \|\eta - \zeta\|_\alpha \quad \text{for all } \eta, \zeta \in X_\alpha. \quad (3.10)$$

**Theorem 3.1** *Under the above conditions (3.8) to (3.10) the generator  $\mathcal{L}$  given in (3.5) is well-defined for  $f \in L(X) \cap C_0(X)$  and generates a Markov semigroup  $(S(t) : t \geq 0)$  on  $L(X)$  which uniquely specifies a ZRP  $(\eta_t : t \geq 0)$ .*

**Proof.** Andjel (1982). The proof includes in particular the statement that  $\eta_0 \in X_\alpha$  implies  $\eta_t \in X_\alpha$  for all  $t \geq 0$ , which follows from showing that the semigroup is contractive, i.e.

$$|S(t)f(\eta) - S(t)f(\zeta)| \leq l(f) e^{3\bar{g}t/(1-\alpha)} \|\eta - \zeta\|_\alpha.$$

**Remarks.**

- Let  $\mu$  be a measure on  $\mathbb{N}^\Lambda$  with density

$$\mu(\eta(x)) \leq C_1 C_2^{|x|} \quad \text{for some } C_1, C_2 > 0 \quad (3.11)$$

(this includes in particular uniformly bounded densities). Then for all  $\alpha < 1/C_1$  we have  $\mu(X_\alpha) = 1$ , so the restricted state space is very large and contains most cases of interest.

- The conditions (3.8) to (3.10) are sufficient but not necessary, in particular (3.8) can be relaxed when looking on regular lattices and imposing a finite range condition on  $p(x, y)$ .

### 3.2 Stationary measures

Let  $(\eta_t : t \geq 0)$  be a (non-degenerate, well defined) ZRP on a lattice  $\Lambda$  with jump probabilities  $p(x, y)$  and jump rates  $g_x$ .

**Lemma 3.2** *There exists a positive harmonic function  $\lambda = (\lambda_x : x \in \Lambda)$  such that*

$$\sum_{y \in \Lambda} p(y, x) \lambda_y = \lambda_x, \quad (3.12)$$

*which is unique up to multiples.*

**Proof.** Existence of non-negative  $\lambda_x$  follows directly from  $p(x, y)$  being the transition probabilities of a random walk on  $\Lambda$ , irreducibility of  $p(x, y)$  implies uniqueness up to multiples and strict positivity.  $\square$

Note that we do not assume  $\lambda$  to be normalizable, which is only the case if the corresponding random walk is positive recurrent. Since (3.12) is homogeneous, every multiple of  $\lambda$  is again a solution. In the following we fix  $\lambda_0 = 1$  (for some lattice site  $0 \in \Lambda$ , say the origin) and denote the one-parameter family of solutions to (3.12) by

$$\{\phi\lambda : \phi \geq 0\}, \quad (3.13)$$

where the parameter  $\phi$  is called the *fugacity*.

**Theorem 3.3** *For each  $\phi \geq 0$ , the product measure  $\nu_\phi$  with marginals*

$$\nu_\phi^x(\eta(x) = n) = \frac{w_x(n)(\phi\lambda_x)^n}{z_x(\phi)} \quad \text{and} \quad w_x(n) = \prod_{k=1}^n \frac{1}{g_x(k)} \quad (3.14)$$

*is stationary, provided that the local normalization (also called **partition function**)*

$$z_x(\phi) = \sum_{n=0}^{\infty} w_x(n)(\phi\lambda_x)^n < \infty \quad \text{for all } x \in \Lambda. \quad (3.15)$$

**Proof.** To simplify notation in the proof we will write

$$\nu_\phi^x(n) := \nu_\phi^x(\eta(x) = n), \quad (3.16)$$

and we will assume that  $\Lambda$  is finite. Our argument can be immediately extended to infinite lattices. First note that using  $w_x(n) = 1 / \prod_{k=1}^n g_x(k)$  we have for all  $n \geq 0$

$$\nu_\phi^x(n+1) = \frac{1}{z_x(\phi)} w_x(n+1)(\phi\lambda_x)^{n+1} = \frac{\phi\lambda_x}{g_x(n+1)} \nu_\phi^x(n). \quad (3.17)$$

We have to show that for all cylinder test functions  $f$

$$\nu_\phi(\mathcal{L}f) = \sum_{\eta \in X} \sum_{x, y \in \Lambda} g_x(\eta(x)) p(x, y) (f(\eta^{x \rightarrow y}) - f(\eta)) \nu_\phi(\eta) = 0, \quad (3.18)$$

which will be done by two changes of variables.

1. For all  $x, y \in \Lambda$  we change variables in the sum over  $\eta$

$$\sum_{\eta \in X} g_x(\eta(x)) p(x, y) f(\eta^{x \rightarrow y}) \nu(\eta) = \sum_{\eta \in X} g_x(\eta(x) + 1) p(x, y) f(\eta) \nu(\eta^{y \rightarrow x}). \quad (3.19)$$

Using (3.17) we have

$$\begin{aligned} \nu_\phi(\eta^{y \rightarrow x}) &= \nu_\phi^x(\eta(x) + 1) \nu_\phi^y(\eta(y) - 1) \prod_{z \neq x, y} \nu_\phi^z(\eta(z)) = \\ &= \frac{\phi\lambda_x}{g_x(\eta(x) + 1)} \nu_\phi^x(\eta(x)) \frac{g_y(\eta(y))}{\phi\lambda_y} \nu_\phi^y(\eta(y)) \prod_{z \neq x, y} \nu_\phi^z(\eta(z)) = \\ &= \nu_\phi(\eta) \frac{\lambda_x g_y(\eta(y))}{\lambda_y g_x(\eta(x))}. \end{aligned} \quad (3.20)$$

Plugging this into (3.18) we get

$$\nu_\phi(\mathcal{L}f) = \sum_{\eta \in X} f(\eta) \nu_\phi(\eta) \sum_{x, y \in \Lambda} \left( g_y(\eta(y)) p(x, y) \frac{\lambda_x}{\lambda_y} - g_x(\eta(x)) p(x, y) \right). \quad (3.21)$$

2. Exchanging summation variables  $x \leftrightarrow y$  in the first part of the above sum we get

$$\nu_\phi(\mathcal{L}f) = \sum_{\eta \in X} f(\eta) \nu_\phi(\eta) \sum_{x \in \Lambda} \frac{g_x(\eta(x))}{\lambda_x} \sum_{y \in \Lambda} (p(y, x) \lambda_y - p(x, y) \lambda_x) = 0, \quad (3.22)$$

since

$$\sum_{y \in \Lambda} (p(y, x) \lambda_y - p(x, y) \lambda_x) = \sum_{y \in \Lambda} (p(y, x) \lambda_y) - \lambda_x = 0. \quad (3.23)$$

Note that terms of the form  $\nu_\phi^y(-1)$  do not appear in the above sums, since  $g_y(0) = 0$ .  $\square$

**Examples.** Take  $\Lambda = \Lambda_L = \mathbb{Z}/L\mathbb{Z}$ ,  $p(x, y) = p \delta_{y, x+1} + q \delta_{y, x-1}$  corresponding to nearest-neighbour jumps on a one-dimensional lattice with periodic boundary conditions. Then we simply have  $\lambda_x = 1$  for all  $x \in \Lambda_L$  as the solution to (3.12).

For the constant jump rates  $g_x(n) = 1$  for all  $n \geq 1$ ,  $x \in \Lambda_L$  the stationary weights are just  $w_x(n) = 1$  for all  $n \geq 0$ <sup>1</sup>. So the stationary product measures  $\nu_\phi$  have geometric marginals

$$\nu_\phi^x(\eta(x) = n) = (1 - \phi) \phi^n \quad \text{since} \quad z_x(\phi) = \sum_{k=0}^{\infty} \phi^k = \frac{1}{1 - \phi}, \quad (3.24)$$

which are well defined for all  $\phi \in [0, 1)$ .

For independent particles with jump rates  $g_x(n) = n$  for all  $x \in \Lambda_L$  we have  $w_x(n) = 1/n!$  and the  $\nu_\phi$  have Poisson marginals

$$\nu_\phi^x(\eta(x) = n) = \frac{\phi^n}{n!} e^{-\phi} \quad \text{since} \quad z_x(\phi) = \sum_{k=0}^{\infty} \frac{\phi^k}{k!} = e^\phi, \quad (3.25)$$

which are well defined for all  $\phi \geq 0$ .

**Remarks.**

- The partition function  $z_x(\phi) = \sum_{n=0}^{\infty} w_x(n) (\phi \lambda_x)^n$  is a power series with radius of convergence

$$r_x = \left( \limsup_{n \rightarrow \infty} w_x(n)^{1/n} \right)^{-1} \quad \text{and so} \quad z_x(\phi) < \infty \quad \text{if} \quad \phi < r_x / \lambda_x. \quad (3.26)$$

If  $g_x^\infty = \lim_{k \rightarrow \infty} g_x(k)$  exists, we have

$$w_x(n)^{1/n} = \left( \prod_{k=1}^n g_x(k)^{-1} \right)^{1/n} = \exp \left( -\frac{1}{n} \sum_{k=1}^n \log g_x(k) \right) \rightarrow 1/g_x^\infty \quad (3.27)$$

as  $n \rightarrow \infty$ , so that  $r_x = g_x^\infty$ .

---

<sup>1</sup>We always use the convention that the empty product  $\prod_{k=1}^0 1/g_x(k) = 1$ .

- The density at site  $x \in \Lambda$  is given by

$$\rho_x(\phi) = \nu_\phi^x(\eta(x)) = \frac{1}{z_x(\phi)} \sum_{k=1}^{\infty} k w_x(k) (\phi \lambda_x)^k. \quad (3.28)$$

Multiplying the coefficients  $w_x(k)$  by  $k$  (or any other polynomial) does not change the radius of convergence of the power series and therefore  $\rho_x(\phi) < \infty$  for all  $\phi < r_x/\lambda_x$ . Furthermore  $\rho_x(0) = 0$  and it can be shown that  $\rho_x(\phi)$  is a monotone increasing function of  $\phi$  (see problem sheet). Note that for  $\phi > r_x/\lambda_x$  the partition function and  $\rho_x(\phi)$  diverge, but for  $\phi = r_x/\lambda_x$  both, convergence or divergence, are possible.

- With Def. 2.3 the expected stationary current across a bond  $(x, y)$  is given by

$$j(x, y) = \nu_\phi^x(g_x) p(x, y) - \nu_\phi^y(g_y) p(y, x), \quad (3.29)$$

and using the form  $w_x(n) = 1/\prod_{k=1}^n g_x(k)$  of the stationary weight we have

$$\begin{aligned} \nu_\phi^x(g_x) &= \frac{1}{z_x(\phi)} \sum_{n=1}^{\infty} g_x(n) w_x(n) (\phi \lambda_x)^n = \\ &= \frac{\phi \lambda_x}{z_x(\phi)} \sum_{n=1}^{\infty} w_x(n-1) (\phi \lambda_x)^{n-1} = \phi \lambda_x. \end{aligned} \quad (3.30)$$

So the current is given by

$$j(x, y) = \phi (\lambda_x p(x, y) - \lambda_y p(y, x)), \quad (3.31)$$

which is proportional to the fugacity  $\phi$  and the stationary probability current of a single random walker (as long as  $\lambda$  can be normalized).

**Examples.** For the above example with  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$ ,  $p(x, y) = p \delta_{y, x+1} + q \delta_{y, x-1}$  and  $g_x(n) = 1$  for  $n \geq 1$ ,  $x \in \Lambda$  the density is of course  $x$ -independent and given by

$$\rho_x(\phi) = \rho(\phi) = (1 - \phi) \sum_{k=1}^{\infty} k \phi^k = \frac{\phi}{1 - \phi} \quad (\text{mean of a geometric}). \quad (3.32)$$

The stationary current  $j(x, x+1) = \phi(p - q)$  for all  $x \in \Lambda_L$ , and as we have seen before in one-dimensional systems it is bond-independent. Using the invertible relation (3.32) we can write the stationary current as a function of the density  $\rho$  analogous to the ASEP in Section 2,

$$j(\rho) = (p - q) \frac{\rho}{1 + \rho}, \quad (3.33)$$

where we use the same letter  $j$  to avoid notational overload.

For independent particles with  $g_x(n) = n$  for all  $x \in \Lambda$ , we get the very simple relation

$$\rho(\phi) = e^{-\phi} \sum_{k=1}^{\infty} k \frac{\phi^k}{k!} = \phi e^{-\phi} \sum_{k=0}^{\infty} \frac{\phi^k}{k!} = \phi \quad (\text{mean of a Poisson}). \quad (3.34)$$

For the current this implies

$$j(\rho) = (p - q)\rho, \quad (3.35)$$

which is to be expected for independent particles.

### 3.3 Equivalence of ensembles and relative entropy

In this section let  $(\eta_t : t \geq 0)$  be a homogeneous ZRP on the lattice  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  with state space  $X_L = \mathbb{N}^{\Lambda_L}$ , jump rates  $g_x(n) \equiv g(n)$  and translation invariant jump probabilities  $p(x, y) = q(y - x)$ . This implies that the stationary product measures  $\nu_\phi$  given in Theorem 3.3 are translation with marginals

$$\nu_\phi^x(\eta(x) = n) = \frac{w(n) \phi^n}{z(\phi)}. \quad (3.36)$$

Analogous to Section 2.1 for exclusion processes, the family of measures

$$\{\nu_\phi^L : \phi \in [0, \phi_c)\} \quad \text{is called} \quad \mathbf{grand-canonical ensemble}, \quad (3.37)$$

where  $\phi_c$  is the radius of convergence of the partition function  $z(\phi)$  (called  $r_x$  in the previous section for more general processes). We further assume that the jump rates are bounded away from 0, i.e.  $g(k) \geq C > 0$  for all  $k > 0$ , which implies that  $w(k) \leq C^{-k}$  and thus  $\phi_c \geq C > 0$  using (3.27). The particle density  $\rho(\phi)$  is characterized uniquely by the fugacity  $\phi$  as given in (3.28).

As noted before the ZRP is irreducible on

$$X_{L,N} = \{\eta \in \mathbb{N}^{\Lambda_L} \mid \Sigma_L(\eta) = N\} \quad (3.38)$$

for all fixed particle numbers  $N \in \mathbb{N}$ . It has a unique stationary measure  $\pi_{L,N}$  on  $X_{L,N}$ , and analogous to the ASEP in Section 2.2 it can be written as a conditional product measure

$$\begin{aligned} \pi_{L,N}(\eta) &= \nu_\phi^L(\eta | X_{L,N}) = \mathbb{1}_{X_{L,N}}(\eta) \frac{\phi^N \prod_x w(\eta(x))}{z(\phi)^L} \frac{z(\phi)^L}{\phi^N \sum_{\eta \in X_{L,N}} \prod_x w(\eta(x))} = \\ &= \frac{\mathbb{1}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x \in \Lambda_L} w(\eta(x)), \end{aligned} \quad (3.39)$$

where we write  $Z_{L,N} = \sum_{\eta \in X_{L,N}} \prod_x w(\eta(x))$  for the *canonical partition function*. The family of measures

$$\{\pi_{L,N} : N \in \mathbb{N}\} \quad \text{is called} \quad \mathbf{canonical ensemble}. \quad (3.40)$$

In general these two ensembles are expected to be 'equivalent' as  $L \rightarrow \infty$ , in vague analogy to the law of large numbers for iid random variables. We will make this precise in the following. To do this we need to quantify the 'distance' of two probability measures.

**Definition 3.2** Let  $\mu_1, \mu_2 \in \mathcal{M}_1(\Omega)$  be two probability measures on a countable space  $\Omega$ . Then the *relative entropy* of  $\mu_1$  w.r.t.  $\mu_2$  is defined as

$$H(\mu_1; \mu_2) = \begin{cases} \mu_1 \left( \log \frac{\mu_1}{\mu_2} \right) = \sum_{\omega \in \Omega} \mu_1(\omega) \log \frac{\mu_1(\omega)}{\mu_2(\omega)} & , \text{ if } \mu_1 \ll \mu_2 \\ \infty & , \text{ if } \mu_1 \not\ll \mu_2 \end{cases}, \quad (3.41)$$

where  $\mu_1 \ll \mu_2$  is a shorthand for  $\mu_2(\omega) = 0 \Rightarrow \mu_1(\omega) = 0$  (called *absolute continuity*).

#### Lemma 3.4 Properties of relative entropy

Let  $\mu_1, \mu_2 \in \mathcal{M}_1(\Omega)$  be two probability measures on a countable space  $\Omega$ .

(i) **Non-negativity:**

$$H(\mu_1; \mu_2) \geq 0 \quad \text{and} \quad H(\mu_1; \mu_2) = 0 \Leftrightarrow \mu_1(\omega) = \mu_2(\omega) \text{ for all } \omega \in \Omega.$$

(ii) **Sub-additivity:**

Suppose  $\Omega = S^\Lambda$  with some local state space  $S \subseteq \mathbb{N}$  and a lattice  $\Lambda$ . Then for  $\Delta \subseteq \Lambda$  and marginals  $\mu_i^\Delta$ ,  $H(\mu_1^\Delta; \mu_2^\Delta)$  is increasing in  $\Delta$  and

$$H(\mu_1; \mu_2) \geq H(\mu_1^\Delta; \mu_2^\Delta) + H(\mu_1^{\Lambda \setminus \Delta}; \mu_2^{\Lambda \setminus \Delta}). \quad (3.42)$$

If  $\mu_1$  and  $\mu_2$  are product measures, then equality holds.

(iii) **Entropy inequality:**

For all bounded  $f \in C_b(\Omega)$  and all  $\epsilon > 0$  we have

$$\mu_1(f) \leq \frac{1}{\epsilon} \left( \log \mu_2(e^{\epsilon f}) + H(\mu_1; \mu_2) \right). \quad (3.43)$$

**Proof.** In the following let  $\mu_1 \ll \mu_2$  and  $h(\omega) = \mu_1(\omega)/\mu_2(\omega) \geq 0$ .

(i) Then

$$H(\mu_1; \mu_2) = \mu_2(h \log h) = \mu_2(\phi(h)) \quad \text{with} \quad \phi(u) := u \log u + 1 - u, \quad (3.44)$$

since  $\mu_2(1 - h) = 1 - \mu_1(1) = 1 - 1 = 0$ . Elementary properties of  $\phi$  are

$$\phi(u) \geq 0 \text{ for } u \geq 0 \quad \text{and} \quad \phi(u) = 0 \Leftrightarrow u = 1, \quad (3.45)$$

which implies that  $H(\mu_1; \mu_2) \geq 0$ . If  $\mu_1 = \mu_2$  the relative entropy obviously vanishes.

On the other hand, if  $H(\mu_1; \mu_2) = 0$  then  $\phi(h(\omega)) = 0$  whenever  $\mu_2(\omega) > 0$ , which implies  $h(\omega) = 1$  and thus  $\mu_1(\omega) = \mu_2(\omega)$ . Since  $\mu_1 \ll \mu_2$  equality also holds when  $\mu_2(\omega) = 0$ .

(ii) For  $\Omega = S^\Lambda$  we fix some  $\Delta \subsetneq \Lambda$  and write  $h(\eta) = \mu_1(\eta)/\mu_2(\eta)$  and  $h^\Delta(\eta(\Delta)) = \mu_1^\Delta(\eta(\Delta))/\mu_2^\Delta(\eta(\Delta))$  for marginal distributions with  $\Delta \subseteq \Lambda_L$ . Then  $h^\Delta$  is given by an expectation conditioned on the sub-configuration  $\eta(\Delta)$  on  $\Delta$ ,

$$h^\Delta(\eta(\Delta)) = \frac{\mu_1^\Delta}{\mu_2^\Delta}(\eta(\Delta)) = \mu_2 \left( \frac{\mu_1}{\mu_2} \middle| \eta(\Delta) \right) = \mu_2(h | \eta(\Delta)). \quad (3.46)$$

Since  $\phi$  is convex we can apply Jensen's inequality to get

$$\phi(h^\Delta(\eta(\Delta))) = \phi \left( \mu_2(h | \eta(\Delta)) \right) \leq \mu_2(\phi(h) | \eta(\Delta)). \quad (3.47)$$

Therefore with  $\mu_2 \left( \mu_2(\phi(h) | \eta(\Delta)) \right) = \mu_2(\phi(h))$  we have

$$H(\mu_1^\Delta; \mu_2^\Delta) = \mu_2(\phi(h^\Delta)) \leq \mu_2(\phi(h)) = H(\mu_1; \mu_2), \quad (3.48)$$

which implies that in general  $H(\mu_1^\Delta; \mu_2^\Delta)$  is increasing in  $\Delta$ .

Using the auxiliary measure  $\nu = \frac{\mu_1^\Delta}{\mu_2^\Delta} \mu_2$  monotonicity in  $\Delta$  implies

$$\begin{aligned} H(\mu_1; \mu_2) - H(\mu_1^\Delta; \mu_2^\Delta) &= \mu_1 \left( \log \frac{\mu_1 \mu_2^\Delta}{\mu_2 \mu_1^\Delta} \right) = \mu_1 \left( \log \frac{\mu_1}{\nu} \right) = H(\mu; \nu) \geq \\ &\geq H(\mu_1^{\Lambda \setminus \Delta}; \nu^{\Lambda \setminus \Delta}) = H(\mu_1^{\Lambda \setminus \Delta}; \mu_2^{\Lambda \setminus \Delta}), \end{aligned} \quad (3.49)$$

since  $\nu^{\Lambda \setminus \Delta} = \mu_2^{\Lambda \setminus \Delta}$  by definition ( $\mu_1^\Delta / \mu_2^\Delta$  does not change  $\mu_2$  on  $\Lambda \setminus \Delta$ ).  
 If  $\mu_1$  and  $\mu_2$  are product measures  $h = \mu_1 / \mu_2$  factorizes, leading to equality.  
 (iii) harder, see e.g. [11], Appendix 1. □

**Remarks.**

- $H(\mu_1; \mu_2)$  is not symmetric and therefore not a metric on  $\mathcal{M}_1(X)$ .
- (i) only holds if  $\mu_1, \mu_2 \in \mathcal{M}_1(X)$  are normalized probability measures, for general distributions in  $\mathcal{M}(X)$  the relative entropy can also be negative.
- $H(\mu_1; \mu_2)$  is a well studied concept from information theory, often also called *Kullback-Leibler divergence* or *information gain*.

**Theorem 3.5** Consider the canonical and grand-canonical ensembles for a homogeneous ZRP as defined above. Then the **specific relative entropy**

$$h_L(\phi) := \frac{1}{L} H(\pi_{L,N}; \nu_\phi^L) \rightarrow 0 \quad (3.50)$$

in the **thermodynamic limit**  $L \rightarrow \infty$  and  $N/L \rightarrow \bar{\rho} \geq 0$ , provided that  $\phi \in [0, \phi_c)$  solves  $\rho(\phi) = \bar{\rho}$ .

**Proof.** First we fix some  $L \geq 0$ . Note that for all  $\eta \in X_L$  and  $\phi > 0$ ,  $\nu_\phi(\eta) > 0$ , so in particular  $\pi_{L,N} \ll \nu_\phi$  and we have

$$h_L(\phi) = \frac{1}{L} \sum_{\eta \in X_{L,N}} \pi_{L,N}(\eta) \log \frac{\pi_{L,N}(\eta)}{\nu_\phi^L(\eta)}. \quad (3.51)$$

Using the form (3.36) and (3.39) of the two measures we get for  $\eta \in X_{L,N}$

$$\frac{\pi_{L,N}(\eta)}{\nu_\phi^L(\eta)} = \frac{\prod_x w(\eta(x))}{Z_{L,N}} \frac{z(\phi)^L}{\prod_x w(\eta(x)) \phi^{\eta(x)}} = \frac{z(\phi)^L}{Z_{L,N} \phi^N}. \quad (3.52)$$

So due to the special form of the ensembles we get the simple expression

$$h_L(\phi) = \frac{1}{L} \sum_{\eta \in X_{L,N}} \pi_{L,N}(\eta) \log \frac{z(\phi)^L}{Z_{L,N} \phi^N} = -\frac{1}{L} \log \frac{Z_{L,N} \phi^N}{z(\phi)^L}. \quad (3.53)$$

Further note that

$$Z_{L,N} = \sum_{\eta \in X_{L,N}} \prod_{x \in \Lambda_L} w(\eta(x)) = \nu_\phi^L(\Sigma_L(\eta) = N) \phi^{-N} z(\phi)^L, \quad (3.54)$$

and thus

$$h_L(\phi) = -\frac{1}{L} \log \nu_\phi^L(\Sigma_L(\eta) = N). \quad (3.55)$$

Since  $\phi < \phi_c$  we have  $\sum_n n^2 w(n) \phi^n < \infty$ . So under  $\nu_\phi$  the  $\eta(x)$  are iidrvs with finite variance and mean  $\nu_\phi^x(\eta(x)) = \rho(\phi) = \bar{\rho}$ . Now taking  $L \rightarrow \infty$  with  $N/L \rightarrow \bar{\rho}$  by the local central limit theorem (see e.g. [28], Chapter 9)

$$\nu_\phi^L(\Sigma_L(\eta) = N) = \nu_\phi^L\left(\sum_{x \in \Lambda_L} \eta(x) = N\right) = O(L^{-1/2}), \quad (3.56)$$

which corresponds to the width  $\sqrt{L}$  of the distribution of a sum of iidrv's. This implies that

$$h_L(\phi) = O\left(\frac{1}{L} \log L\right) \rightarrow 0 \quad \text{as } L \rightarrow \infty. \quad (3.57)$$

□

Note that this convergence result only holds if  $\bar{\rho}$  is in the range of the function  $\rho(\phi)$  for  $\phi \in [0, \phi_c)$ . Whenever this is not the case the system exhibits an interesting phase transition which is discussed in detail in the next section.

**Corollary 3.6** *Let  $f \in C_0(X)$  be a cylinder test function with  $\nu_\phi(e^{\epsilon f}) < \infty$  for some  $\epsilon > 0$ . Then with  $\nu_\phi$  being the product measure on the whole lattice,*

$$\mu_{L,N}(f) \rightarrow \nu_\phi(f) \quad \text{as } L \rightarrow \infty, \quad (3.58)$$

*provided that  $\phi \in [0, \phi_c)$  solves  $\rho(\phi) = \bar{\rho} = \lim_{L \rightarrow \infty} N/L$ .*

**Proof.** Let  $\Delta \subseteq \Lambda_L$  be the finite range of dependence of the cylinder function  $f \in C_0(X)$ . Then we can plug  $f - \nu_\phi^\Delta(f)$  and  $\nu_\phi^\Delta(f) - f$  in the entropy inequality (3.43) to show that

$$|\pi_{L,N}(f) - \nu_\phi(f)| \leq H(\pi_{L,N}^\Delta; \nu_\phi^\Delta). \quad (3.59)$$

This involves extending the inequality to unbounded functions  $f$  with finite exponential moments and a standard  $\epsilon - \delta$  argument. It is rather lengthy and we do not present this here, for a reference see e.g. [25], Lemma 3.1.

Then sub-additivity (Lemma 3.4(ii)) gives

$$H(\pi_{L,N}^\Delta; \nu_\phi^\Delta) \leq \frac{|\Delta|}{L} H(\pi_{L,N}; \nu_\phi^L) = |\Delta| h_L(\phi) \rightarrow 0 \quad (3.60)$$

as  $L \rightarrow \infty$  which implies the statement. □

### Remarks.

- The above corollary implies e.g. convergence of the test function  $f(\eta) = \eta(x)$ , since for all  $\phi < \phi_c$

$$\sum_{n=0}^{\infty} e^{\epsilon n} w(n) \phi^n < \infty \quad \text{for } e^\epsilon \phi < \phi_c, \text{ i.e. } \epsilon < \log \frac{\phi_c}{\phi}. \quad (3.61)$$

So  $\pi_{L,N}(\eta(x)) = N/L \rightarrow \nu_\phi(\eta(x)) = \rho(\phi)$ , which is not very surprising since  $\phi$  is chosen to match the limiting density  $\bar{\rho}$ .

- The function  $f(\eta) = \eta(x)^2$  corresponding to the second moment is not covered by the above result, since  $e^{\epsilon n^2}$  grows too fast with  $n$  for all  $\epsilon > 0$ . However, convergence can be extended to functions  $f \in L^2(\nu_\phi)$  (with considerable technical effort, see e.g. appendix of [11]). Since  $\phi < \phi_c$  leads to an exponential decay of  $w(n)\phi^n$ , this extension includes all polynomial correlation functions.

### 3.4 Phase separation and condensation

Since ZRPs are bosonic lattice gases, they exhibit a condensation transition under certain conditions which is similar to Bose-Einstein condensation for bosons. For more details and other applications and related results to this section see [27] and references therein. As in the previous section we consider a homogeneous ZRP on the lattice  $\Lambda_L = \mathbb{Z}/L\mathbb{Z}$  with jump rates  $g(n)$  bounded away from 0 for  $n > 0$  and translation invariant jump probabilities  $p(x, y) = q(y - x)$ .

**Definition 3.3** Let  $\rho(\phi) = \nu_\phi(\eta(x))$  be the density of the grand-canonical product measure  $\nu_\phi$  and  $\phi_c \in [0, \infty]$  be the radius of convergence of the partition function  $z(\phi)$ . Then we define the *critical density*

$$\rho_c = \lim_{\phi \nearrow \phi_c} \rho(\phi) \in [0, \infty]. \quad (3.62)$$

$\rho_c$  can take the value  $\infty$ , as we have seen above for the example

$$g(k) = 1 - \delta_{k,0} \quad \Rightarrow \quad \rho(\phi) = \frac{\phi}{1 - \phi} \nearrow \infty \quad \text{as } \phi \nearrow \phi_c = 1. \quad (3.63)$$

In fact, this is the 'usual' situation since it implies that the function  $\rho(\phi)$  is a bijection and there exists a grand-canonical stationary measure for all densities  $\rho \geq 0$ .

To have an example with  $\rho_c < \infty$  we need

$$\sum_{n=0}^{\infty} n w(n) \phi_c^n < \infty, \quad (3.64)$$

i.e. the power series has to converge **at** the radius of convergence  $\phi_c$ . Therefore  $w(n)\phi_c^n$  has to decay sub-exponentially (by definition of  $\phi_c$ ), but fast enough for the sum to converge. A generic example is a power law decay

$$w(n) \phi_c^n \simeq n^{-b} \quad \text{as } n \rightarrow \infty \quad \text{with } b > 2. \quad (3.65)$$

Since we have the explicit formula  $w(n) = \prod_{k=1}^n g(k)^{-1}$  this implies for the jump rates

$$g(n) = \frac{w(n-1)}{w(n)} \simeq \frac{(n-1)^{-b} \phi_c^{-(n-1)}}{n^{-b} \phi_c^{-n}} = \phi_c (1 - 1/n)^{-b} \simeq \phi_c (1 + b/n) \quad (3.66)$$

to leading order. Such a ZRP with rates

$$g(n) = 1 + b/n \quad \text{with } \phi_c = 1 \quad \text{and } w(n) \simeq \Gamma(1+b) n^{-b} \quad (3.67)$$

was introduced [26]. For this model  $\rho_c$  can be computed explicitly,

$$\rho_c = \frac{1}{b-2} < \infty \quad \text{for } b > 2. \quad (3.68)$$

The interesting question is now, what happens to the equivalence of ensembles in the limit  $L \rightarrow \infty$  with  $N/L \rightarrow \bar{\rho} > \rho_c$ ?

**Theorem 3.7** Consider the canonical  $\pi_{L,N}$  and the grand-canonical measures  $\nu_\phi^L$  of a homogeneous ZRP, for which we assume that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \log g(k) \in \mathbb{R} \quad \text{exists.} \quad (3.69)$$

Then

$$h_L(\phi) := \frac{1}{L} H(\pi_{L,N}; \nu_\phi^L) \rightarrow 0 \quad \text{as } L \rightarrow \infty \quad \text{and } N/L \rightarrow \bar{\rho} \geq 0, \quad (3.70)$$

provided that for  $\bar{\rho} \leq \rho_c$ ,  $\phi \in [0, \phi_c]$  solves  $\rho(\phi) = \rho$  (**sub-critical case**) and for  $\bar{\rho} > \rho_c$ ,  $\phi = \phi_c$  (**super-critical case**).

**Proof.** Analogous to the proof of Theorem 3.5 we have

$$h_L(\phi) = -\frac{1}{L} \log \nu_\phi^L(\Sigma_L(\eta) = N), \quad (3.71)$$

and for  $\bar{\rho} \leq \rho_c$  or  $\rho_c = \infty$  this implies the result as before.

For  $\bar{\rho} > \rho_c$ ,  $\sum_{x \in \Lambda_L} \eta(x) = N$  is a large deviation event, and to get an upper bound on (3.71) we need a lower bound on its probability under the critical measure  $\nu_{\phi_c}^L$ .

$$\begin{aligned} \nu_{\phi_c}^L \left( \sum_{x \in \Lambda_L} \eta(x) = N \right) &\geq \\ &\geq \nu_{\phi_c}^1(\eta(1) = N - [\rho_c(L-1)]) \nu_{\phi_c}^{\Lambda_L \setminus \{1\}} \left( \sum_{x \in \Lambda_L \setminus \{1\}} \eta(x) = [\rho_c(L-1)] \right), \end{aligned} \quad (3.72)$$

which corresponds to putting an extensive amount of particles on a single lattice site (we arbitrarily chose 1), and distributing an amount which is typical under  $\nu_{\phi_c}$  on the remaining sites.

The second term can be treated by local limit theorems analogous to the previous result\* (see remark below). Since  $\phi_c$  is the radius of convergence of the partition function,  $\nu_{\phi_c}^1$  has a subexponential tail, i.e.

$$\frac{1}{L} \log \nu_{\phi_c}^1(\eta(1) = N - [\rho_c(L-1)]) \rightarrow 0 \quad \text{as } L \rightarrow \infty, \quad (3.73)$$

since  $N - [\rho_c(L-1)] \simeq (\bar{\rho} - \rho_c)L \rightarrow \infty$  for  $\bar{\rho} > \rho_c$ . The fact that this holds not only along a subsequence but the limit really exists, is guaranteed by assumption (3.69) using

$$\log \nu_{\phi_c}^1(\eta(1) = n) = n \log(\phi_c w(n)^{1/n}) - \log z(\phi_c) \quad (3.74)$$

and (3.27). Plugging these results for (3.72) into (3.71) we get  $h_L(\phi_c) \rightarrow 0$  for  $\bar{\rho} > \rho_c$ .  $\square$

### Remarks.

- Existence of the (*Cesàro*) limit in (3.69) is a very weak assumption, it is certainly fulfilled if  $g(k)$  has a limit as  $k \rightarrow \infty$  as in our example above. It only excludes pathological cases where  $g(k)$  has an exponentially diverging subsequence.
- \*For  $b > 3$  the  $\eta(x)$  are iidrvs with finite variance and the second term in (3.72) is of order  $1/\sqrt{L}$ . For  $2 < b \leq 3$  the variance is infinite and the sum of  $\eta(x)$  has a non-normal limit distribution. Using adapted local limit theorems (see also [28], Chapter 9), the second term can still be bounded below by terms of order  $1/L$  for all  $b > 2$ .
- Corollary 3.6 still applies, but note that in the super-critical case  $\nu_{\phi_c}(e^{\epsilon \eta(x)}) = \infty$  for all  $\epsilon > 0$  due to sub-exponential tails. So the test function  $f(\eta) = \eta(x)$  is not included in the result, which is to be expected, since for  $\rho > \rho_c$

$$\pi_{L,N}(\eta(x)) = N/L \rightarrow \rho > \rho_c = \nu_{\phi_c}(\eta(x)). \quad (3.75)$$

### Interpretation.

- Elements  $\nu_\phi$  of the grand-canonical ensemble are also called *fluid phases*. For  $\rho > \rho_c$  the ensemble

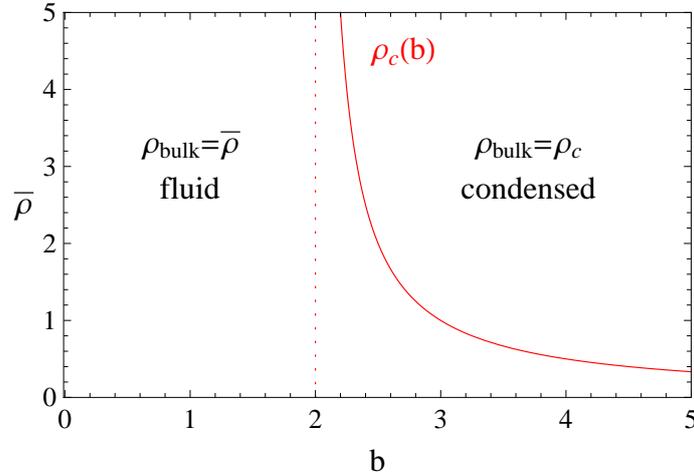
$$\{\nu_\phi : \phi \in [0, \phi_c]\} \quad \text{has density range} \quad [0, \rho_c], \quad (3.76)$$

and there are no fluid phases with density  $\rho > \rho_c$ .

- The limiting distribution in any finite fixed volume  $\Delta$  is given by the fluid phase  $\nu_{\phi_c}^\Delta$  with density is  $\rho_c$ . Therefore for large systems the *excess mass*  $(\rho - \rho_c)L$  concentrates in a region with vanishing volume fraction (volume  $o(L)$ ), the so-called *condensed phase*. This phenomenon is called *phase separation* in general, and since one of the phases covers only a vanishing fraction of the system this particular form of phase separation is called *condensation*.
- It can be shown (see [29]) that in fact the condensed phase concentrates on a *single* lattice site, i.e. for  $\rho > \rho_c$  we have a law of large numbers for the maximal occupation number in the canonical ensemble,

$$\pi_{L,N} \left( \left| \frac{1}{L} \max_{x \in \Lambda_L} \eta(x) - (\rho - \rho_c) \right| > \epsilon \right) \rightarrow 0 \quad \text{as } L \rightarrow \infty \quad \text{for all } \epsilon > 0. \quad (3.77)$$

For the above example with  $g(k) = 1 + b/k$ ,  $k > 0$  and  $\rho_c(b) = 1/(b - 2)$  these results can be summarized in the following phase diagram.



The axes are given by the system parameters  $b$  and the density  $\bar{\rho} = \lim_{L \rightarrow \infty} N/L$ . As order parameter we took the limiting *bulk density*  $\rho_{bulk} := \nu_\phi(\eta(x))$ , where  $\nu_\phi$  is the limit measure of Theorem 3.7. This leads to

$$\rho_{bulk} = \begin{cases} \bar{\rho} & , \bar{\rho} \leq \rho_c \\ \rho_c & , \bar{\rho} > \rho_c \end{cases}, \quad (3.78)$$

corresponding to two phase regions which we call *fluid* and *condensed*.  $\rho_{bulk}$  is continuous across the phase transition line (red), and therefore condensation is a continuous phase transition w.r.t. the order parameter  $\rho_{bulk}$ .

## 4 The contact process

The lattice  $\Lambda$ , an arbitrary countable set, is endowed with a graph structure by a directed edge set  $E \subseteq \Lambda \times \Lambda$ . We assume that  $(\Lambda, E)$  is strongly connected, i.e. for all  $x, y \in \Lambda$  there exists a directed path of edges connecting  $x$  to  $y$ . The state space of the contact process (CP) is  $X = \{0, 1\}^\Lambda$  and the generator is

$$\mathcal{L}f(\eta) = \sum_{z \in \Lambda} \left( \eta(z) + \lambda(1 - \eta(z)) \sum_{y \sim z} \eta(y) \right) (f(\eta^z) - f(\eta)) , \quad (4.1)$$

where  $y \sim x$  if  $(y, x) \in E$ . Infected sites ( $\eta(x) = 1$ ) recover independently with rate 1, and infect neighbouring sites independently with rate  $\lambda > 0$ .

### 4.1 Mean-field rate equations

Choosing  $f(\eta) = \eta(x)$ , denoting by  $\mu_t = \mu_0 S(t)$  the distribution at time  $t$  and writing  $\rho(x, t) = \mu_t(\eta(x)) \in [0, 1]$  for the density, we get from the forward equation (1.46)

$$\frac{d}{dt} \rho(x, t) = \mu_t(\mathcal{L}f) = -\rho(x, t) + \lambda \sum_{y \sim x} \mu_t(\eta(y)(1 - \eta(x))) . \quad (4.2)$$

This follows from plugging  $f(\eta) = \eta(x)$  and  $f(\eta^x) = 1 - \eta(x)$  into (4.1), which leads to

$$\begin{aligned} \mathcal{L}f(\eta) &= \eta(x)(1 - \eta(x)) + \lambda(1 - \eta(x))^2 \sum_{y \sim x} \eta(y) - \eta(x)^2 - \lambda \eta(x)(1 - \eta(x)) \sum_{y \sim x} \eta(y) \\ &= -\eta(x) + \lambda(1 - \eta(x)) \sum_{y \sim x} \eta(y) , \end{aligned} \quad (4.3)$$

since  $\eta(x) \in \{0, 1\}$  leads to simplifications  $\eta(x)(1 - \eta(x)) = 0$  and  $\eta(x)^2 = \eta(x)$ . Note that only the term  $z = x$  in the sum in (4.1) contributes.

So the time evolution of the first moment  $\rho(t)$  involves second moments and is not a closed equation, similar to what we have seen for the ASEP in Section 2. The simplest way to close these equations is called the *mean-field assumption*:

$$\mu_t(\eta(y)(1 - \eta(x))) = \mu_t(\eta(y))\mu_t(1 - \eta(x)) = \rho(y, t)(1 - \rho(x, t)) , \quad (4.4)$$

i.e.  $\mu_t$  is assumed to be a product measure and the  $\eta(x)$  to be independent. If the graph  $(\Lambda, E)$  is translation invariant, e.g. a regular lattice such as  $\mathbb{Z}^d$  or  $(\mathbb{Z}/L\mathbb{Z})^d$  or homogeneous trees, and the initial distribution  $\mu_0$  is as well, the system is homogeneous and we have the additional identity  $\rho(x, t) \equiv \rho(t)$  for all  $x \in \Lambda$ . Using this and the mean-field assumption in (4.2) we get the *mean-field rate equation* for the CP

$$\frac{d}{dt} \rho(t) = -\rho(t) + m\lambda\rho(t)(1 - \rho(t)) , \quad (4.5)$$

where  $m$  is the coordination number or vertex degree of the lattice  $\Lambda$ , i.e. the number of neighbours of a lattice site, such as  $m = 2d$  for  $d$ -dimensional cubic lattices.

**Remarks.**

- Of course there is no reason why the mean-field assumption should be correct, in fact it is known to be false (see later sections). However, it turns out that for high coordination number the replacement

$$\mu_t \left( \sum_{y \sim x} \eta(y)(1 - \eta(x)) \right) \approx \sum_{y \sim x} \rho_t(y)(1 - \rho_t(x)) \quad (4.6)$$

leads to quantitatively good predictions. Due to a 'law of large numbers'-effect  $\sum_{y \sim x} \eta(y)$  can be replaced by its expected value when the number  $m$  of terms is large. For example this is the case for high dimensional cubic lattices with  $m = 2d$ , and it can even be shown that mean-field results are 'exact' as long as  $d > 4$ . The highest dimension for which the mean-field assumption is not exact is often referred to as the *upper critical dimension* in the physics literature.

- Another situation with high coordination number is when the lattice  $\Lambda$  is actually a complete graph, i.e.  $E = \Lambda_L \times \Lambda_L$ . Here it can be shown that (4.5) holds rigorously with  $m + L$  for  $\rho(t) = \frac{1}{L} \sum_{x \in \Lambda_L} \rho(x, t)$ .
- For low dimensions/coordination numbers the mean-field assumption still is useful to get a first idea of the critical behaviour of the system, since it typically easy to derive and analyze. In most cases quantitative predictions are wrong (such as location of phase boundaries and critical exponents), but qualitative features are often predicted correctly (such as the number of phase regions or existence of critical points).

#### Analysis of the rate equation.

The long-time behaviour of solutions to an equation of the form  $\frac{d}{dt}\rho(t) = f(\rho(t))$  is given by stationary points of the right-hand side  $f(\rho) = 0$ . In our case for (4.5) these are given by

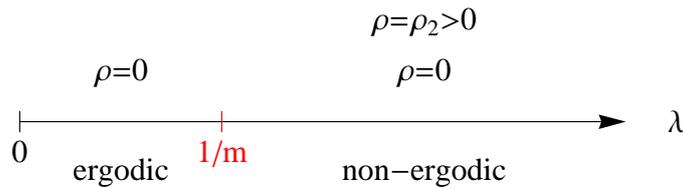
$$0 = -\rho + m\lambda\rho(1 - \rho) = -m\lambda\rho^2 + (m\lambda - 1)\rho, \quad (4.7)$$

which are the roots of a downward parabola, given by  $\rho_1 = 0$  and  $\rho_2 = 1 - 1/(m\lambda)$ .

$\rho \equiv \rho_1 = 0$  is always a stationary solution to the equation, corresponding to the absorbing state  $\eta = 0$  of the CP, called the *inactive phase*. For  $m\lambda > 1$  there is a second stationary density  $\rho_2 = 1 - 1/(m\lambda) \in (0, 1)$  called the *active phase*. The domains of attraction of these stationary points are determined by the sign of  $f(\rho)$ , and  $\rho_i$  is locally stable if  $f'(\rho_i) < 0$ . In summary we have

$$\begin{aligned} f'(0) = m\lambda - 1 &\Rightarrow \rho = 0 && \begin{array}{l} \text{stable for } m\lambda \leq 1 \\ \text{unstable for } m\lambda > 1 \end{array} \\ f'(\rho_2) = 1 - m\lambda &\Rightarrow \rho = \rho_2 && \begin{array}{l} \notin (0, 1] \text{ for } m\lambda \leq 1 \\ \text{stable for } m\lambda > 1 \end{array} \end{aligned} \quad (4.8)$$

which leads to the following mean-field prediction of the phase diagram of the CP with the *critical value*  $\lambda_c = 1/m$ .



As opposed to previous sections the diagram is one-dimensional, since the number of particles in the CP is not conserved and  $\lambda$  is the only system parameter. The two phase regions can be characterized by ergodicity of the infinite system, as is explained below.

**Remarks.**

- The mean-field rate equation does not take into account fluctuations. Since the CP is irreducible on  $X \setminus \{0\}$ , on a finite lattice the states in the active phase are transient and the CP is ergodic with unique stationary measure  $\mu = \delta_0$ .  
However, if the infection rate  $\lambda$  is large enough and we start the system in the active phase (e.g.  $\eta_0(x) = 1$  for all  $x$ ), it remains active for a (random) time with mean of the order  $\exp(CL)$  where  $L$  is the size of the lattice. If  $L$  is large it takes the system very long to reach its stationary distribution and the active phase is said to be *metastable* (see e.g. [5], Chapter I.3).
- The lifetime of the active phase diverges for infinite lattice size. Therefore infinite systems exhibit a truly stationary active phase if  $\lambda$  is large enough. The system is no longer ergodic since it has two stationary distributions,  $\delta_0$  corresponding to the absorbing state (inactive phase) and  $\mu$  corresponding to the active phase (more details on  $\mu$  follow later).
- On  $\mathbb{Z}$  ( $d = 1$ ) precise numerical estimates (and rigorous bounds) show that  $\lambda_c = 1.64893$ , which is quite far from the mean-field value  $1/m = 1/2$  predicted by (4.5). Nevertheless, the qualitative prediction of a phase transitions turns out to be true. Comparing to the first remark it is actually not surprising that mean-field underestimates the critical value, since even for  $\lambda > 1/2$  the system can still die out due to fluctuations. Clearly  $\lambda_c$  should decrease with  $m$  since the total infection rate of one infected site is actually  $m\lambda$ , and in fact the numerical estimate for  $\mathbb{Z}^2$  is 0.4119 (MF prediction  $1/m = 0.25$ ).

**4.2 Stochastic monotonicity and coupling**

In this section we introduce a powerful technique which can be used to get rigorous results on the contact process. Let  $X = S^\Lambda$  be the state space of a particle system with  $S \subseteq \mathbb{N}$  and  $\Lambda$  some arbitrary discrete lattice.  $X$  is a partially ordered set, given by

$$\eta \leq \zeta \quad \text{if} \quad \eta(x) \leq \zeta(x) \quad \text{for all } x \in \Lambda. \tag{4.9}$$

**Definition 4.1** A function  $f \in C(X)$  is *increasing* if

$$\eta \leq \zeta \quad \text{implies} \quad f(\eta) \leq f(\zeta). \tag{4.10}$$

This leads to the concept of *stochastic monotonicity* for probability measures  $\mu_1, \mu_2$  on  $X$ :

$$\mu_1 \leq \mu_2 \quad \text{provided that} \quad \mu_1(f) \leq \mu_2(f) \quad \text{for all increasing } f \in C(X). \tag{4.11}$$

This definition is quite hard to work with, and the best way to understand and use stochastic monotonicity is in terms of *couplings*.

**Definition 4.2** A coupling of two measures  $\mu_1, \mu_2 \in \mathcal{M}_1(X)$  is a measure  $\mu$  on the product state space  $X \times X$  of pair configurations  $\eta = (\eta^1, \eta^2)$ , such that the marginals for  $i = 1, 2$  are

$$\mu^i = \mu_i \quad \text{i.e.} \quad \mu(\{\eta : \eta^i \in A\}) = \mu_i(A) \quad \text{for all measurable } A \subseteq X. \tag{4.12}$$

**Remark.** In other words, a coupling means constructing the random variables  $\eta^1(\omega)$  and  $\eta^2(\omega)$  on the same probability space  $(\Omega, \mathcal{A}, \mathbb{P})$ , such that

$$\mathbb{P}(\{\omega : \eta^i(\omega) \in A\}) = \mu_i(A) \quad \text{for all measurable } A \subseteq X. \quad (4.13)$$

**Theorem 4.1 (Strassen)** Suppose  $\mu_1, \mu_2 \in \mathcal{M}_1(X)$ . Then  $\mu_1 \leq \mu_2$  if and only if there exists a coupling  $\mu \in \mathcal{M}_1(X \times X)$  such that

$$\mu(\{\eta : \eta^1 \leq \eta^2\}) = 1 \quad (\eta^1 \leq \eta^2 \mu - a.s.). \quad (4.14)$$

**Proof.**  $\Leftarrow$ : Suppose such a coupling  $\mu$  exists. If  $f \in C(X)$  is increasing then  $f(\eta^1) \leq f(\eta^2)$   $\mu - a.s.$  and writing  $\pi^i : X \times X \rightarrow X$  for the projection on the  $i$ -th coordinate  $\pi^i(\eta) = \eta^i$ , we have

$$\mu_1(f) = \mu(f \circ \pi^1) \leq \mu(f \circ \pi^2) = \mu_2(f), \quad (4.15)$$

so that  $\mu_1 \leq \mu_2$ .

$\Rightarrow$ : involves a construction of the coupling on a probability space, see e.g. Theorem 2.4, p. 72 [5]  $\square$

**Example.** Let  $\nu_{\rho_1}, \nu_{\rho_2}$  be product measures on  $X = \{0, 1\}^\Lambda$  with  $\rho_1 \leq \rho_2$ . Then for each  $i = 1, 2$  the  $\eta^i(x)$  are iid  $Be(\rho_i)$  random variables. We construct a (so-called *maximal*) coupling  $\mu$  on  $X \times X$  that concentrates on configurations  $\eta^1 \leq \eta^2$ . Let  $\Omega_x = (0, 1)$  and  $\mathbb{P}_x = U(0, 1)$  be the uniform measure independently for each  $x \in \Lambda$ . Then define

$$\eta^i(x)(\omega) := \begin{cases} 1 & , \omega_x \leq \rho_i \\ 0 & , \omega_x > \rho_i \end{cases}, \quad (4.16)$$

which implies that  $\eta^1(x)(\omega) \leq \eta^2(x)(\omega)$  for all  $\omega \in \Omega$  and  $x \in \Lambda$ . Taking the product over all lattice sites with  $\mathbb{P} = \prod_x \mathbb{P}_x$ , we can define a coupling measure on  $X \times X$  by

$$\mu := \mathbb{P} \circ \eta^{-1} \quad \text{i.e.} \quad \mu(A) = \mathbb{P}(\{\omega : \eta(\omega) \in A\}) \quad \text{for all } A \in X \times X, \quad (4.17)$$

and we have  $\eta^1 \leq \eta^2 \mu - a.s.$ . Therefore the theorem implies  $\nu_{\rho_1} \leq \nu_{\rho_2}$ .

In practice, to sample from  $\mu$  (i.e. choose a coupled pair of configurations  $\eta^1 \leq \eta^2$ ), first fix  $\eta^1$  by choosing iid  $Be(\rho_1)$  variables. Then under the coupling measure  $\eta^1(x) = 1$  implies  $\eta^2(x) = 1$ , which fixes  $\eta^2$  on those sites. On the remaining empty sites, choose iid  $Be(\frac{\rho_2 - \rho_1}{1 - \rho_1})$  variables. Then the  $\eta^2(x)$  are independent and since  $\mu(\eta^1(x) = 1) = \nu_{\rho_1}(\eta^1(x) = 1) = \rho_1$  we have

$$\mu(\eta^2(x) = 1) = \rho_1 + (1 - \rho_1) \frac{\rho_2 - \rho_1}{1 - \rho_1} = \rho_2, \quad (4.18)$$

so  $\eta^2 \sim \nu_{\rho_2}$  has the right marginal.

The idea of monotonicity and coupling can be extended to processes.

**Definition 4.3** Consider an IPS on  $X$  with generator  $(S(t) : t \geq 0)$ . The process is *attractive* or *monotone* if

$$f \text{ increasing} \quad \Rightarrow \quad S(t)f \text{ increasing for all } t \geq 0, \quad (4.19)$$

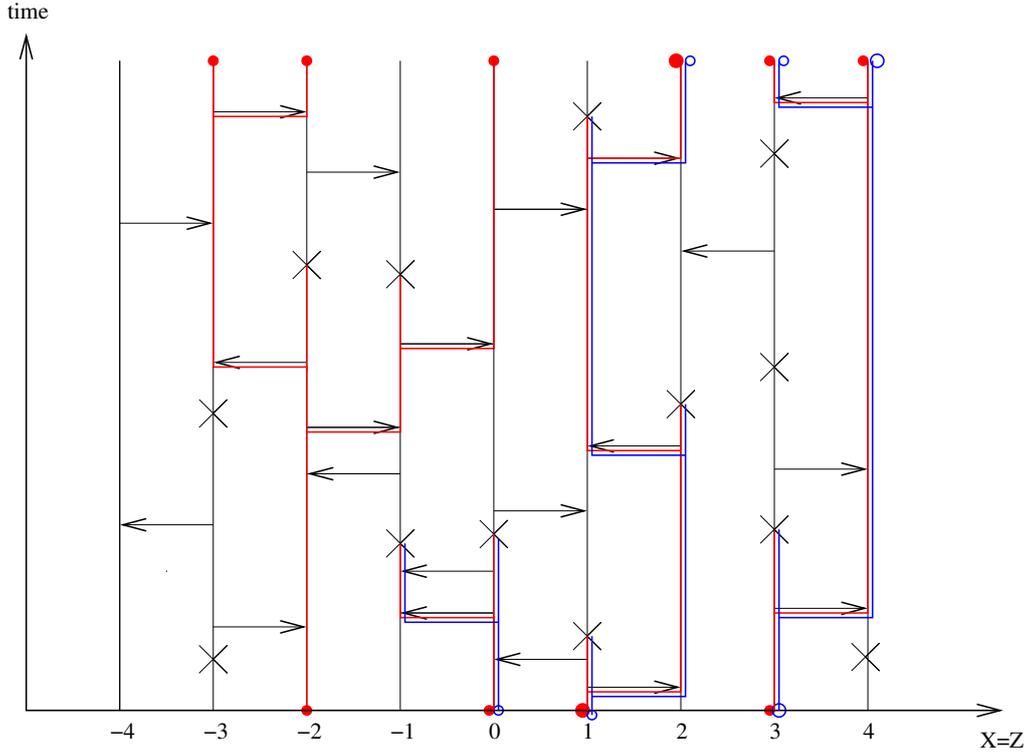
or equivalently

$$\mu_1 \leq \mu_2 \quad \Rightarrow \quad \mu_1 S(t) \leq \mu_2 S(t) \quad \text{for all } t \geq 0. \quad (4.20)$$

Let  $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{M}_1(D[0, \infty))$  be the path space measures of two IPS  $(\eta_t^1 : t \geq 0)$  and  $(\eta_t^2; t \geq 0)$ . Then a *coupling* of the processes is given by a Markov process  $((\eta_t^1, \eta_t^2) : t \geq 0)$  on  $X \times X$  with measure  $\mathbb{P} \sim \mathcal{M}_1(D[0, \infty) \times D[0, \infty))$ , having marginal processes  $(\eta_t^i : t \geq 0) \sim \mathbb{P}_i$ , i.e.  $\mathbb{P}^i = \mathbb{P}_i$ .

**Lemma 4.2** *The contact process is attractive.*

**Proof.** We couple two contact processes  $(\eta_t^1 : t \geq 0)$  (shown red) and  $(\eta_t^2; t \geq 0)$  (shown blue) using a graphical construction.



Both processes use the same realization of infection and recovery processes  $\rightarrow, \leftarrow$  and  $\times$ , and the initial conditions fulfill  $\eta_0^2 \leq \eta_0^1$ . Then by inspection of the coupling construction this immediately implies that  $\eta_t^2 \leq \eta_t^1$  for all  $t \geq 0$  (example shown above). Therefore we have for all  $f \in C(X)$ ,

$$S(t)f(\eta_0^2) = \mathbb{E}^{\eta_0^2}(f(\eta_t^2)) \leq \mathbb{E}^{\eta_0^1}(f(\eta_t^1)) = S(t)f(\eta_0^1), \quad (4.21)$$

and since this holds for all ordered initial conditions the CP is attractive as given in Def. 4.3.  $\square$

More generally it can be shown that:

**Proposition 4.3** *A general spin system on  $\{0, 1\}^\Lambda$  with generator*

$$\mathcal{L}f(\eta) = \sum_{x \in \Lambda} c(x, \eta)(f(\eta^x) - f(\eta)) \quad (4.22)$$

*is attractive if and only if the jump rates (spin flip rates) fulfill*

$$\eta \leq \zeta \quad \text{implies} \quad \begin{cases} c(x, \eta) \leq c(x, \zeta) & , \text{ if } \eta(x) = \zeta(x) = 0 \\ c(x, \eta) \geq c(x, \zeta) & , \text{ if } \eta(x) = \zeta(x) = 1 \end{cases} . \quad (4.23)$$

**Proof.** Suppose the spin system is attractive, i.e.  $f$  increasing implies  $S(t)f$  increasing for all  $t \geq 0$ . Since  $f(\eta) = \eta(x)$  is increasing and in  $C_0(X)$  we have

$$\mathcal{L}f(\eta) = \lim_{t \searrow 0} \frac{S(t)f(\eta) - f(\eta)}{t}, \quad (4.24)$$

and for all  $\eta \leq \zeta$  with  $\eta(x) = \zeta(x)$

$$\mathcal{L}f(\eta) - \mathcal{L}f(\zeta) = \lim_{t \searrow 0} \frac{S(t)f(\eta) - S(t)f(\zeta) + \eta(x) - \zeta(x)}{t} \leq 0. \quad (4.25)$$

Therefore  $\mathcal{L}f(\eta) \leq \mathcal{L}f(\zeta)$  and since

$$\mathcal{L}f(\eta) = c(x, \eta)(1 - 2\eta(x)) \quad (4.26)$$

this implies 4.23.

The other direction involves a more general version of the coupling given in the proof of Lemma 4.2 above, see e.g. Theorem 2.2, p. 134 [5].  $\square$

**Remark.** Property (4.23) asserts that 0 is more likely to flip to 1 in a environment of more 1s ( $\zeta \geq \eta$ ), and vice versa. That means that local occupation numbers 'attract' one another, explaining the term 'attractive' for such particle systems.

#### Lemma 4.4 Monotonicity in $\lambda$

Let  $(\eta_t^\lambda : t \geq 0)$  and  $(\eta_t^{\lambda'} : t \geq 0)$  be two CPs with infection rates  $\lambda \leq \lambda'$ . Then

$$\mu^\lambda \leq \mu^{\lambda'} \quad \text{implies} \quad \mu^\lambda S_\lambda(t) \leq \mu^{\lambda'} S_{\lambda'}(t) \quad \text{for all } t > 0, \quad (4.27)$$

i.e. there exists a coupling such that

$$\eta_0^\lambda \leq \eta_0^{\lambda'} \quad \text{and} \quad \eta_t^\lambda \leq \eta_t^{\lambda'} \quad \text{for all } t > 0. \quad (4.28)$$

**Proof.** By Strassen's Theorem,  $\mu^\lambda \leq \mu^{\lambda'}$  implies existence of a coupling  $\mu \in \mathcal{M}_1(X \times X)$  such that  $\eta_0^\lambda \leq \eta_0^{\lambda'} \mu - a.s.$ . Suppose first that  $\eta_0^\lambda = \eta_0^{\lambda'}$  and couple the processes  $(\eta_t^\lambda : t \geq 0)$  and  $(\eta_t^{\lambda'} : t \geq 0)$  by using coupled infection processes  $PP(\lambda)$  and  $PP(\lambda) + PP(\lambda' - \lambda) \sim PP(\lambda')$  in the graphical construction. Then clearly  $\eta_t^\lambda \leq \eta_t^{\lambda'}$  for all  $t > 0$ . Now by attractivity of the process  $(\eta_t^\lambda : t \geq 0)$  this also holds for initial conditions  $\eta_0^\lambda \leq \eta_0^{\lambda'}$ .  $\square$

### 4.3 Invariant measures and critical values

Consider a CP with infection rate  $\lambda$  on some connected graph  $(\Lambda, E)$  and let  $\delta_0$  be the point mass on the empty configuration and  $\delta_1$  on the full configuration  $\eta(x) = 1, x \in \Lambda$ . Since  $\eta \equiv 0$  is absorbing,  $\delta_0$  is stationary.

**Proposition 4.5** For all  $0 \leq s \leq t$  we have

$$\delta_1 S(t) \leq \delta_1 S(s), \quad \bar{\nu}_\lambda = \lim_{t \rightarrow \infty} \delta_1 S(t) \quad \text{exists and} \quad \bar{\nu}_\lambda \in \mathcal{I}_e. \quad (4.29)$$

$\bar{\nu}_\lambda$  is called the **upper invariant measure**, and we have  $\delta_0 \leq \mu \leq \bar{\nu}_\lambda$  for all  $\mu \in \mathcal{I}$ . Furthermore,  $\lambda < \lambda'$  implies  $\bar{\nu}_\lambda \leq \bar{\nu}_{\lambda'}$ , and for each  $x \in \Lambda$

$$\rho_x(\lambda) := \bar{\nu}_\lambda(\eta(x)) \quad \text{is monotone increasing in } \lambda. \quad (4.30)$$

**Proof.** Since  $\delta_1$  is maximal on  $X$  we have

$$\delta_1 \geq \delta_1 S(t-s) \quad \text{for all } 0 \leq s \leq t. \quad (4.31)$$

By attractivity of the CP and the Markov property this implies

$$\delta_1 S(s) \geq \delta_1 S(t-s)S(s) = \delta_1 S(t). \quad (4.32)$$

Therefore  $\delta_1 S(t)$  is a monotone sequence, and by compactness of  $\mathcal{M}_1(X)$  (in the topology of weak convergence) the limit exists and is stationary by Theorem 1.9(b). Furthermore  $\delta_0 \leq \pi \leq \delta_1$  for all  $\pi \in \mathcal{M}_1$ . Every stationary measure  $\mu \in \mathcal{I}$  can be written as  $\mu = \lim_{t \rightarrow \infty} \pi S(t)$  for some  $\pi \in \mathcal{M}_1$ , so by attractivity

$$\delta_0 S(t) \leq \pi S(t) \leq \delta_1 S(t) \quad \text{and after } t \rightarrow \infty, \quad \delta_0 \leq \mu \leq \bar{\nu}_\lambda. \quad (4.33)$$

Suppose that  $\bar{\nu}_\lambda \in \mathcal{I}$  is not extremal, i.e.  $\bar{\nu}_\lambda = \alpha \mu_1 + (1-\alpha)\mu_2$  for  $\mu_1, \mu_2 \in \mathcal{I}$  and  $\alpha \in (0, 1)$ . Then  $\mu_1, \mu_2 \leq \bar{\nu}_\lambda$ , so for all increasing  $f \in C(X)$  we have  $\mu_1(f), \mu_2(f) \leq \bar{\nu}_\lambda(f)$ . Suppose now that  $\mu_1(f) < \bar{\nu}_\lambda(f)$ , then

$$\alpha \mu_1(f) + (1-\alpha)\mu_2(f) < \alpha \bar{\nu}_\lambda(f) + (1-\alpha)\bar{\nu}_\lambda(f) = \bar{\nu}_\lambda(f) \quad (4.34)$$

in contradiction to the assumption. So  $\mu_1(f) = \mu_2(f) = \bar{\nu}_\lambda(f)$  for all increasing  $f \in C(X)$ , and thus  $\mu_1 = \mu_2 = \bar{\nu}_\lambda$  and  $\bar{\nu}_\lambda \in \mathcal{I}_e$ .

By monotonicity in  $\lambda$  we have for all  $t \geq 0$

$$\delta_1 S^\lambda(t) \leq \delta_1 S^{\lambda'}(t), \quad (4.35)$$

provided that  $\lambda \leq \lambda'$ , which implies  $\bar{\nu}_\lambda \leq \bar{\nu}_{\lambda'}$ . Since  $\eta(x)$  is increasing this also holds for the corresponding densities.  $\square$

On a finite lattice  $\eta \equiv 0$  can be reached in finite time from any other configuration, and since  $\eta \equiv 0$  is absorbing this implies

$$\mu S(t) \rightarrow \delta_0 \quad \text{as } t \rightarrow \infty \quad \text{for all } \mu \in \mathcal{M}_1(X). \quad (4.36)$$

This holds in particular for  $\mu = \delta_1$ , and thus the upper invariant measure is  $\bar{\nu}_\lambda = \delta_0$  and the CP is ergodic for all  $\lambda \geq 0$ . On the other hand, on an infinite lattice it might be possible that  $\bar{\nu}_\lambda \neq \delta_0$  and the mean-field prediction of an active phase is correct. It turns out that this is indeed the case for high enough infection rate  $\lambda$  as we will see below.

**Definition 4.4** Denote by

$$\alpha_\eta := \mathbb{P}^\eta(\eta_t \neq 0 \text{ for all } t \geq 0) \quad (4.37)$$

the *survival probability* with initial configuration  $\eta \in X$ . For each  $x \in \Lambda$  denote by  $\xi_x \in X$  the configuration with  $\xi_x(y) = \delta_{y,x}$  having a single infection at  $x$ . The CP  $(\eta_t : t \geq 0)$  is said to *die out* if  $\alpha_{\xi_x} = 0$  for some  $x \in \Lambda$ , otherwise it is said to *survive*.

Note that condition (4.4) actually does not depend on the lattice site  $x$ , since  $\Lambda$  is connected and therefore the CP is irreducible on  $X \setminus \{0\}$ .

**Proposition 4.6** *If the CP dies out for infection rate  $\lambda' > 0$ , then it dies out for all  $\lambda \in [0, \lambda']$ . The **critical value**  $\lambda_c \in [0, \infty]$  is then given by*

$$\lambda_c := \sup \{ \lambda \geq 0 : \text{CP with infection rate } \lambda \text{ dies out} \}. \quad (4.38)$$

**Proof.** Monotonicity in  $\lambda$  of the CP (Lemma 4.4) and  $\eta_0^\lambda = \eta_0^{\lambda'}$  imply that if  $(\eta_t^{\lambda'} : t \geq 0)$  dies out so does  $(\eta_t^\lambda : t \geq 0)$ .

Since the CP with  $\lambda = 0$  certainly dies out, the supremum  $\lambda_c$  is well defined in  $[0, \infty]$ .  $\square$

**Proposition 4.7** *Analogous to above for any  $A \subseteq \Lambda$  write  $\xi_A \in X$  for  $\xi_A(y) = \mathbb{1}_A(y)$ . Then the survival probability is*

$$\alpha_{\xi_A} = \mathbb{P}^{\xi_A}(\eta_t \neq 0 \text{ for all } t \geq 0) = \bar{\nu}_\lambda(\{\xi_B : B \cap A \neq \emptyset\}), \quad (4.39)$$

and for  $\lambda < \lambda_c$  we have  $\bar{\nu}_\lambda = \delta_0$  for  $\lambda > \lambda_c$ ,  $\bar{\nu}_\lambda \neq \delta_0$ .

**Proof.** The result is based on the following *duality property* of the CP. For all  $A, B \subseteq \Lambda$  where one of them, say  $A$  is finite, we have

$$\mathbb{P}^{\xi_A}(\eta_t(x) = 1 \text{ for some } x \in B) = \mathbb{P}^{\xi_B}(\eta_t(x) = 1 \text{ for some } x \in A). \quad (4.40)$$

For a proof of this see e.g. [9] Theorem VI.1.7. Now choosing  $B = \Lambda$  we have  $\xi_B(x) = 1$  for all  $x \in \Lambda$  and

$$\mathbb{P}^{\xi_A}(\eta_t \neq 0) = \mathbb{P}^{\delta_1}(\eta_t(x) = 1 \text{ for some } x \in A). \quad (4.41)$$

Taking the limit  $t \rightarrow \infty$  implies the first statement. For  $\lambda < \lambda_c$  the process dies out with probability 1 for all initial configurations  $\xi_x$  and thus with  $A = \{x\}$  in (4.39) we have

$$\bar{\nu}_\lambda(\eta(x) = 1) = \bar{\nu}_\lambda(\eta(x)) = \rho_x(\lambda) = 0 \quad \text{for all } x \in \Lambda, \quad (4.42)$$

which implies that  $\bar{\nu}_\lambda = \delta_0$ . For  $\lambda > \lambda_c$  the process survives, and thus (4.42) has non-zero value and  $\bar{\nu}_\lambda \neq \delta_0$ .  $\square$

**Remark.** Note that Prop. 4.7 implies in particular that the density

$$\rho_x(\lambda) = \bar{\nu}_\lambda(\eta(x)) = \mathbb{P}^{\xi_x}(\eta_t \neq 0 \text{ for all } t \geq 0) \quad (4.43)$$

is equal to the survival probability.

Our results so far imply that there is a well defined critical value  $\lambda_c \in [0, \infty]$  such that the CP dies out and  $\bar{\nu}_\lambda = \delta_0$  for  $\lambda < \lambda_c$ , and the CP survives and  $\bar{\nu}_\lambda \neq \delta_0$  for  $\lambda > \lambda_c$ . On a finite lattice we have discussed above that  $\lambda_c = \infty$ . The crucial question on infinite lattices is now whether  $\lambda_c$  is *non-trivial*, i.e.  $\lambda_c \in (0, \infty)$ . Certainly the value of  $\lambda_c$  will depend on the lattice  $\Lambda$  but at least one can derive a quite general lower bound.

Let  $(\eta_t : t \geq 0)$  be the CP with infection rate  $\lambda$  on a connected graph  $(\Lambda, E)$ . Consider the auxiliary process  $(\zeta_t : t \geq 0)$  on the same graph with state space  $X = \mathbb{N}^\Lambda$  and generator

$$\mathcal{L}f(\zeta) = \sum_{x \in \Lambda} \left( \eta(x) (f(\zeta^{-x}) - f(\zeta)) + \lambda \sum_{y \sim x} \zeta(y) (f(\zeta^{+x}) - f(\zeta)) \right), \quad (4.44)$$

where we write  $\zeta^{\pm x}(y) = \begin{cases} \zeta(y) \pm 1 & , y = x \\ \zeta(y) & , y \neq x \end{cases}$ . In this process particles independently create new particles at connected sites with rate  $\lambda$  and die independently with rate 1, so the number of particles per site can be larger than 1. We couple this process to a CP  $(\eta_t : t \geq 0)$  by using the same Poisson processes  $PP(\lambda)$  and  $PP(1)$  for infection/creation and death/recovery in the graphical

construction. If for the auxiliary process  $\zeta_t > 1$ , we use independent creation and death processes for the extra particles. This construction implies that the CP is *dominated* by the  $\zeta$ -process, i.e.

$$\eta_0 \leq \zeta_0 \quad \Rightarrow \quad \eta_t \leq \zeta_t \quad \text{for all } t \geq 0. \quad (4.45)$$

Therefore if  $(\zeta_t : t \geq 0)$  dies out then the CP dies out as well. Now let  $m$  be the maximal vertex degree of the graph  $(\Lambda, E)$ . Then the number of particles in the  $\zeta$ -process is dominated by a Markov chain  $N(t)$  on the state space  $\mathbb{N}$  with transition rates

$$c(n, n+1) = nm\lambda \quad \text{for } n \geq 0, \quad c(n, n-1) = n \quad \text{for } n \geq 1. \quad (4.46)$$

All the particles independently create new particles at rate  $m\lambda$  and die at rate 1. Again there exists an obvious coupling such that

$$\sum_{x \in \Lambda} \zeta_t(x) \leq N(t) \quad \text{for all } t \geq 0. \quad (4.47)$$

$N(t)$  is a well-known birth-death chain with absorbing state  $n = 0$ , and dies out with probability 1 if and only if  $m\lambda \leq 1$ . For  $m\lambda > 1$  the average  $\mathbb{E}(N(t))$  is monotone increasing and the process can survive with positive probability.

**Proposition 4.8 Lower bound for  $\lambda_c$**  Consider a CP on a connected graph  $(\Lambda, E)$  with maximal vertex degree  $m$ . Then  $\lambda_c \geq 1/m$ .

**Proof.** With initial condition  $\xi_x$  as in Definition 4.4 and using the above coupling the number of active sites in the CP is dominated by the birth-death chain

$$\sum_{x \in \Lambda} \eta_t(x) \leq N(t) \quad \text{with} \quad N(0) = 1. \quad (4.48)$$

Therefore  $\lambda \leq 1/m$  implies that the CP dies out and thus  $\lambda_c \geq 1/m$ .  $\square$

Note that the lower bound coincides with the mean-field prediction  $\lambda_c = 1/m = 1/(2d)$  of Section 4.1. To get an upper bound on  $\lambda_c$  is in general harder. In the following we will concentrate on  $\Lambda = \mathbb{Z}^d$  and only give a small part of the proof.

#### 4.4 Results for $\Lambda = \mathbb{Z}^d$

Consider the CP on the regular lattice  $\Lambda = \mathbb{Z}^d$ .

**Theorem 4.9** For the critical value  $\lambda_c(d)$  of a CP on the lattice  $\Lambda = \mathbb{Z}^d$  we have

$$\frac{1}{2d} \leq \lambda_c(d) \leq \frac{2}{d} \quad \text{for all } d \geq 1. \quad (4.49)$$

**Proof.** The lower bound is given by Prop. 4.8, for the proof of  $\lambda_c(1) \leq 2$  see Theorem VI.1.33 in [9]. For higher dimensions the required inequality  $\lambda_c(d) \leq \lambda_c(1)/d$  follows from

$$\mathbb{P}^{\xi_x}(\eta_t^d \neq 0) \geq \mathbb{P}^{\xi_x}(\eta_t^1 \neq 0), \quad t \geq 0, \quad (4.50)$$

where  $(\eta_t^d : t \geq 0)$  is the  $d$ -dimensional CP with rate  $\lambda$ , and  $(\eta_t^1 : t \geq 0)$  is a 1-dimensional CP with rate  $d\lambda$ . We show this by coupling the two processes such that for each  $y \in \mathbb{Z}$

$$\eta_t^1(y) = 1 \quad \text{implies} \quad \eta_t^d(x) = 1 \quad \text{for some } x \text{ such that } \pi_d(x) = y, \quad (4.51)$$

where for all  $x \in \mathbb{Z}^d$  we denote

$$\pi_d(x) = \pi_d(x_1, \dots, x_d) = x_1 + \dots + x_d \in \mathbb{Z}. \quad (4.52)$$

Suppose that  $A \subseteq \mathbb{Z}^d$  and  $B \subseteq \mathbb{Z}$  are finite and such that

$$B \subseteq \pi_d(A) = \{\pi_d(x) : x \in A\}, \quad (4.53)$$

i.e. for each  $y \in B$  there is (at least) one  $x \in A$  such that  $y = \pi_d(x)$ . Choose one of these  $\bar{x}$ , and associate its  $PP(1)$  death process with site  $y$ . Also, for all of the  $2d$  neighbours of  $\bar{x}$  we have

$$x \sim \bar{x} \quad \text{implies} \quad \pi_d(x) = y \pm 1 \sim y. \quad (4.54)$$

Now associate the infection processes  $PP(\lambda)$  pointing towards  $\bar{x}$  from all its neighbours with infections at  $y$ , which leads to a net infection rate of  $d\lambda$  from each of the two neighbours  $y \pm 1$ . Note that all other deaths and infections in the  $d$ -dimensional CP that would correspond to  $y$  are not used in the coupling. With this construction both marginal processes  $(\eta_t^1 : t \geq 0)$  and  $(\eta_t^d : t \geq 0)$  have the right law, and clearly (4.51) is fulfilled, which finishes the proof.  $\square$

Using more involved techniques than we do here, lower and upper bound can be improved significantly, depending on the dimension  $d$ . Further it can be shown that

$$d\lambda_c(d) \rightarrow \frac{1}{2} \quad \text{as } d \rightarrow \infty, \quad (4.55)$$

supporting the physicist wisdom that 'mean-field theory is exact in high dimensions'.

#### Theorem 4.10 Complete convergence

Consider the CP on  $\lambda = \mathbb{Z}^d$ . For every  $\eta \in X$  as  $t \rightarrow \infty$  we have

$$\delta_\eta S(t) \rightarrow \alpha_\eta \bar{\nu}_\lambda + (1 - \alpha_\eta) \delta_0 \quad \text{weakly (locally)}, \quad (4.56)$$

where  $\alpha_\eta = \mathbb{P}^\eta(\eta_t \neq 0 \text{ for all } t \geq 0)$  is the survival probability.

**Proof.** See e.g. [5], Theorem I.2.27.

#### Remark.

Taking the expected value w.r.t. an initial distribution  $\mu$  in (4.56) we get weak convergence of

$$\mu S(t) \rightarrow \mu(\alpha_\eta) \bar{\nu}_\lambda + (1 - \mu(\alpha_\eta)) \delta_0. \quad (4.57)$$

This holds in particular for all stationary  $\mu \in \mathcal{M}_1(X)$ , and therefore every stationary distribution is a convex combination of  $\delta_0$  and  $\bar{\nu}_\lambda$  and we have

$$\mathcal{I}_e = \{\delta_0, \bar{\nu}_\lambda\}. \quad (4.58)$$

#### Theorem 4.11 Extinction time

Suppose  $\lambda > \lambda_c$  and for the CP  $(\eta_t : t \geq 0)$  let

$$\tau := \inf\{t \geq 0 : \eta_t = 0\} \quad (4.59)$$

be the **extinction time** of the process. Then there exists  $\epsilon > 0$  such that for every initial condition  $\eta_0 = \eta \in X$

$$\mathbb{P}^\eta(\tau < \infty) \leq e^{-\epsilon|\eta|} \quad \text{where} \quad |\eta| = \sum_{x \in \Lambda} \eta(x). \quad (4.60)$$

**Proof.** see [5], Theorem I.2.30

Note that this implies that the supercritical CP can only die out with positive probability if the initial condition is finite  $|\eta| < \infty$ . If, however,  $\mu \in \mathcal{M}_1(X)$  is translation invariant and  $\mu(\eta(x)) > 0$ , then we have  $\mu(|\eta| = \infty) = 1$ , and therefore

$$\mathbb{P}^\eta(\tau = \infty) = \alpha_\eta = 1 \quad (4.61)$$

and the process survives with probability 1. With Theorem 4.10 this implies

$$\mu S(t) \rightarrow \bar{\nu}_\lambda \quad \text{as } t \rightarrow \infty. \quad (4.62)$$

**Theorem 4.12** *The critical contact process dies out.*

**Proof.** see [5], Theorem I.2.25

This implies that the density

$$\rho(\lambda) = \bar{\nu}_\lambda(\eta(x)) = \mathbb{P}^{\xi^x}(\eta_t \neq 0 \text{ for all } t \geq 0) \quad (4.63)$$

which is independent of  $x$  due to translation invariance, is a continuous function of  $\lambda$ . By Proposition 4.5 it is also monotone increasing, for  $\lambda > \lambda_c$  and vanishes for  $\lambda < \lambda_c$  by Proposition 4.7. In particular, to leading order the behaviour at the critical point is given by

$$\rho(\lambda) \sim C(\lambda - \lambda_c)^\beta \quad (4.64)$$

for some exponent  $\beta > 0$ . The only rigorous bound is  $\beta \leq 1$ , and our mean-field result from section 4.1 predicts  $\lambda_c = 1/(2d)$  and for  $\lambda \geq \lambda_c$  similar we have to leading order

$$\rho(\lambda) = 1 - \frac{1}{2d\lambda} = 1 - \frac{1}{2d\lambda_c} \left(1 + \frac{\lambda - \lambda_c}{\lambda_c}\right)^{-1} \simeq \frac{\lambda - \lambda_c}{\lambda_c}, \quad (4.65)$$

which implies  $\beta = 1$ . In fact numerical estimates give values  $\beta \approx 0.28$  ( $d = 1$ ),  $0.58$  ( $d = 2$ ),  $0.81$  ( $d = 3$ ), and for  $d \geq 4$  the mean-field value  $\beta = 1$  should be 'exact'.

The CP has also been analyzed on other regular lattices, in particular homogeneous trees  $T^d$  (see e.g. Chapter I.4 in [5]). In this case the critical behaviour turns out to be more complicated, there exists a second critical value  $\lambda_2 > \lambda_c$  and complete convergence in the sense of Theorem 4.10 only holds outside the interval  $[\lambda_c, \lambda_2]$ . Inside this interval there exist infinitely many extremal invariant measures and the infection survives globally but dies out locally.

## 4.5 Duality

**Definition 4.5** Consider two independent Markov processes  $(\eta_t : t \geq 0)$  on  $X$  and  $(\xi_t : t \geq 0)$  on  $\tilde{X}$  with corresponding path measures  $\mathbb{P}^\eta$  and  $\tilde{\mathbb{P}}^\xi$ .  $(\xi_t : t \geq 0)$  is the *dual* of  $(\eta_t : t \geq 0)$  with *duality function*  $D : X \times \tilde{X} \rightarrow \mathbb{R}$  if

$$\mathbb{E}^\eta D(\eta_t, \xi) = \tilde{\mathbb{E}}^\xi D(\eta, \xi_t) \quad \text{for all } \eta \in X \text{ and } \xi \in \tilde{X}. \quad (4.66)$$

An equivalent formulation using semigroups  $(S(t) : t \geq 0)$  and  $(\tilde{S}(t) : t \geq 0)$  is

$$S(t) D(\cdot, \xi)(\eta) = \tilde{S}(t) D(\eta, \cdot)(\xi) \quad \text{for all } \eta \in X \text{ and } \xi \in \tilde{X}. \quad (4.67)$$

If  $X = \tilde{X}$  and  $\mathbb{P}^\eta = \tilde{\mathbb{P}}^\eta$  for all  $\eta \in X$ ,  $(\eta_t : t \geq 0)$  is called *self-dual*.

**Proposition 4.13** Consider the processes  $(\eta_t : t \geq 0)$  on  $X$  with generator  $\mathcal{L}$  and  $(\xi_t : t \geq 0)$  on  $\tilde{X}$  with generator  $\tilde{\mathcal{L}}$ . The processes are dual duality function  $D : X \times \tilde{X} \rightarrow \mathbb{R}$  if and only if

$$\mathcal{L}D(\cdot, \xi)(\eta) = \mathcal{L}D(\eta, \cdot)(\xi) \quad \text{for all } \eta \in X \text{ and } \xi \in \tilde{X}. \quad (4.68)$$

This holds provided that  $\mathcal{L}D(\cdot, \xi)$  and  $\tilde{D}(\eta, \cdot)$  are well defined for all  $\eta \in X$  and  $\xi \in \tilde{X}$ .

**Proof.** Assume duality of  $(\eta_t : t \geq 0)$  and  $(\xi_t : t \geq 0)$ . Then

$$\frac{1}{t}(S(t)D(\cdot, \xi)(\eta) - D(\eta, \xi)) = \frac{1}{t}(\tilde{S}(t)D(\eta, \cdot)(\xi) - D(\eta, \xi)) \quad (4.69)$$

for all  $t > 0$ . Taking the limit  $t \searrow 0$  implies (4.68) using the definition (1.44) of the generator. By the Hille-Yosida Theorem 1.6 the reverse follows from taking the limit  $n \rightarrow \infty$  in the identity

$$\left( Id + \frac{t}{n}\mathcal{L} \right)^n D(\cdot, \xi)(\eta) = \left( Id + \frac{t}{n}\tilde{\mathcal{L}} \right)^n D(\eta, \cdot)(\xi), \quad (4.70)$$

which holds for all  $n \in \mathbb{N}$  by induction over  $n$ .  $\square$

**Remarks.**

- $\mathcal{L}D$  and  $\tilde{\mathcal{L}}D$  are well defined e.g. if  $(\eta_t : t \geq 0)$  and  $(\xi_t : t \geq 0)$  are Markov chains with countable state space. If they are IPS with state spaces  $X$  and  $\tilde{X}$  then  $D(\cdot, \xi)$  and  $D(\eta, \cdot)$  should be cylinder functions for all  $\eta \in X$  and  $\xi \in \tilde{X}$ .
- Duality is a symmetric relation, i.e. if  $(\eta_t : t \geq 0)$  is dual to  $(\xi_t : t \geq 0)$  then  $(\xi_t : t \geq 0)$  is dual to  $(\eta_t : t \geq 0)$  with the same duality function modulo coordinate permutation.

**Proposition 4.14** The CP with  $X = \{0, 1\}^\Lambda$ ,  $\Lambda$  connected, is self-dual.

**Proof.** For  $\eta \in X$  and  $A \subseteq \Lambda$  finite define

$$D(\eta, A) := \prod_{x \in A} (1 - \eta(x)) = \begin{cases} 1, & \text{if } \eta \equiv 0 \text{ on } A \\ 0, & \text{otherwise} \end{cases}. \quad (4.71)$$

Then, using  $D(\eta, A) = (1 - \eta(x))D(\eta, A \setminus \{x\})$  for  $x \in A$ , we have

$$D(\eta^x, A) - D(\eta, A) = \begin{cases} D(\eta, A \setminus \{x\}) & , x \in A, \eta(x) = 1 \\ -D(\eta, A) & , x \in A, \eta(x) = 0 \\ 0 & , x \notin A \end{cases}. \quad (4.72)$$

This implies for the generator of the contact process  $(\eta_t : t \geq 0)$

$$\begin{aligned} \mathcal{L}D(\cdot, A)(\eta) &= \sum_{x \in \Lambda} \left( \eta(x) + \lambda(1 - \eta(x)) \sum_{y \sim x} \eta(y) \right) (D(\eta^x, A) - D(\eta, A)) = \\ &= \sum_{x \in A} \left( \eta(x) D(\eta, A \setminus \{x\}) - \lambda \sum_{y \sim x} \eta(y) (1 - \eta(x)) D(\eta, A) \right). \end{aligned} \quad (4.73)$$

Using  $(1 - \eta(x))D(\eta, A) = (1 - \eta(x))D(\eta, A \setminus \{x\}) = D(\eta, A)$  for  $x \in A$  and writing  $\eta(x) = \eta(x) - 1 + 1$  we get

$$\begin{aligned} \mathcal{L}D(\cdot, A)(\eta) &= \sum_{x \in A} \left( D(\eta, A \setminus \{x\}) - D(\eta, A) + \lambda \sum_{y \sim x} \left( D(\eta, A \cup \{y\}) - D(\eta, A) \right) \right) = \\ &=: \tilde{\mathcal{L}}D(\eta, \cdot)(A). \end{aligned} \quad (4.74)$$

Now  $\tilde{\mathcal{L}}$  is a generator on  $\tilde{X} = \{A \subseteq \Lambda \text{ finite}\}$  with transitions

$$\begin{aligned} A &\rightarrow A \setminus \{x\} && \text{at rate } 1, \text{ if } x \in A, \\ A &\rightarrow A \cup \{y\} && \text{at rate } \lambda |\{x \in A : x \sim y\}|, \text{ if } y \notin A. \end{aligned} \quad (4.75)$$

If we identify  $A = \{x : \tilde{\eta}(x) = 1\}$  to be the set of infected sites of a process  $(\tilde{\eta}_t : t \geq 0)$ , then this is again a CP on  $X$  with infection rate  $\lambda$ .  $\square$

**Remark.** It is often convenient to describe a CP  $(\eta_t : t \geq 0)$  also in terms of the set of infections  $(A_t : t \geq 0)$ . We use the same notation for the path measures  $\mathbb{P}$  to indicate that we really have the same process only in different notation. In that sense (4.71) is a duality function for the CP and we have

$$\mathbb{E}^\eta D(\cdot, A) = \mathbb{P}^\eta(\eta_t \equiv 0 \text{ on } A) = \mathbb{P}^A(\eta \equiv 0 \text{ on } A_t) = \mathbb{E}^A D(\eta, \cdot). \quad (4.76)$$

Note that this is the relation we used in the proof of Proposition 4.7 in slightly different notation.

**Proposition 4.15** *Let  $(\eta_t : t \geq 0)$  on  $X$  be dual to  $(\xi_t : t \geq 0)$  on  $\tilde{X}$  w.r.t.  $D : X \times \tilde{X} \rightarrow \mathbb{R}$ . If  $T : C(X) \rightarrow C(X)$  is a simple symmetry or a conservation law for  $(\eta_t : t \geq 0)$  according to Propositions 2.3 and 2.5, then*

$$D'(\eta, \xi) = (T D(\cdot, \xi))(\eta) \quad (4.77)$$

*is also a duality function.*

**Proof.** For a symmetry  $T$  we have  $S(t)T = TS(t)$  for all  $t \geq 0$ , so

$$S(t) D'(\cdot, \xi)(\eta) = S(t) T D(\cdot, \xi)(\eta) = T S(t) D(\cdot, \xi)(\eta) = T(\tilde{S}(t) D(\cdot, \cdot)(\xi))(\eta). \quad (4.78)$$

Now, if  $T$  is a simple symmetry with  $Tf = f \circ \tau$ ,  $\tau : X \rightarrow X$  for all  $f \in C(X)$ , we have

$$T(\tilde{S}(t) D(\cdot, \cdot)(\xi))(\eta) = (\tilde{S}(t) D(\cdot, \cdot)(\xi))(\tau\eta) = \tilde{S}(t) D(\tau\eta, \cdot)(\xi) = \tilde{S}(t) D'(\eta, \cdot)(\xi). \quad (4.79)$$

If  $T$  is a conservation law with  $Tf = gf$  for all  $f \in C(x)$  and some  $g \in C(X)$ ,

$$T(\tilde{S}(t) D(\cdot, \cdot)(\xi))(\eta) = g(\eta) \tilde{S}(t) D(\eta, \cdot)(\xi) = \tilde{S}(t) g(\eta) D(\eta, \cdot)(\xi) = \tilde{S}(t) D'(\eta, \cdot)(\xi), \quad (4.80)$$

since  $g(\eta)$  is a constant under  $\tilde{S}(t)$ , and the latter is a linear operator.  $\square$

**Remarks.**

- Of course it is possible that  $D' = TD = D$ . For example, translation invariance is a symmetry of the CP on  $\Lambda = \mathbb{Z}^d$ , and the duality function  $D$  in (4.71) is translation invariant. But the linear voter model (see Definition 1.4) has a particle-hole symmetry, which can generate two different duality functions (see example sheet).
- The result of Proposition 4.15 holds for all symmetries for which the commutation relation  $T\tilde{S}(t)D = \tilde{S}(t)D$  holds. As seen in the proof this holds for general duality functions  $D$  as long as we restrict to simple symmetries or conservation laws. For more general symmetries regularity assumptions on  $D$  are necessary. Even though  $T$  and  $\tilde{S}(t)$  act on different arguments of  $D(\eta, \xi)$ , they do not necessarily commute in general, like e.g. partial derivatives of a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  only commute if  $f$  is differentiable.

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