1 Let $L \subset \mathbb{P}^n$ be an $(n-1)$-dimensional linear subspace, $X \subset L$ an irreducible closed variety and y a point in $\mathbb{P}^n \setminus L$. Join y to all points $x \in X$ by lines, and denote by Y the set of points lying on all these lines, that is, the cone over X with vertex y. Prove that Y is an irreducible projective variety and $\dim Y = \dim X + 1$.

2 Let $X \subset \mathbb{A}^3$ be the reducible curve whose components are the 3 coordinate axes. Prove that the ideal $\mathfrak{A}_X$ cannot be generated by 2 elements.

3 Let $X \subset \mathbb{P}^2$ be the reducible 0-dimensional variety consisting of 3 points not lying on a line. Prove that the ideal $\mathfrak{A}_X$ cannot be generated by 2 elements.

4 Prove that any finite set $S \subset \mathbb{A}^2$ can be defined by two equations. [Hint: Choose the coordinates x, y in $\mathbb{A}^2$ in such a way that all points of S have different x coordinates; then show how to define S by the two equations $y = f(x)$, $\prod(x - \alpha_i) = 0$, where $f(x)$ is a polynomial.]

5 Prove that any finite set of points $S \subset \mathbb{P}^2$ can be defined by two equations.

6 Let $X \subset \mathbb{A}^3$ be an algebraic curve, and x, y, z coordinates in $\mathbb{A}^3$; suppose that X does not contain a line parallel to the z-axis. Prove that there exists a nonzero polynomial $f(x, y)$ vanishing at all points of X. Prove that all such polynomials form a principal ideal $(g(x, y))$, and that the curve $g(x, y) = 0$ in $\mathbb{A}^2$ is the closure of the projection of X onto the (x, y)-plane parallel to the z-axis.

7 We use the notation of Exercise 6. Suppose that $h(x, y, z) = g_0(x, y)z^n + \cdots + g_n(x, y)$ is the irreducible polynomial of smallest positive degree in z contained in the ideal $\mathfrak{A}_X$. Prove that if $f \in \mathfrak{A}_X$ has degree m as a polynomial in z, then we can write $fg_0^m = hU + v(x, y)$, where $v(x, y)$ is divisible by $g(x, y)$. Deduce that the equation $h = g = 0$ defines a reducible curve consisting of X together with a finite number of lines parallel to the x-axis, defined by $g_0(x, y) = g(x, y) = 0$.

8 Use Exercises 6–7 to prove that any curve $X \subset \mathbb{A}^3$ can be defined by 3 equations.

9 By analogy with Exercises 6–8, prove that any curve $X \subset \mathbb{P}^3$ can be defined by 3 equations.

10 Let $F_0(x_0, \dots, x_n), \dots, F_n(x_0, \dots, x_n)$ be forms of degree $m_0, \dots, m_n$ and consider the system of $n + 1$ equations in $n + 1$ variables $F_0(x) = \cdots = F_n(x) = 0$. Write Γ for the subset of $\prod_{i=0}^n \mathbb{P}^{\nu_{n,m_i}} \times \mathbb{P}^n$ (where $\nu_{n,m} = \binom{n+m}{m} - 1$) defined by

$$\Gamma = \{(F_0, \dots, F_n, x) \mid F$$

By considering the two projection maps $\varphi : \Gamma \to \prod_i \mathbb{P}^{\nu_{n,m_i}}$ and $\psi : \Gamma \to \mathbb{P}^n$, prove that $\dim \Gamma = \dim \varphi(\Gamma) = \sum_i \nu_{n,m_i} - 1$. Deduce from this that there exists a polynomial $R = R(F_0, \dots, F_n)$ in the coefficients of the forms $F_0, \dots, F_n$ such that $R = 0$ is a necessary and sufficient condition for the system of $n+1$ equations in $n+1$ variables to have a nonzero solution. What is the polynomial R if the forms $F_0, \dots, F_n$ are linear?

11 Prove that the Plücker hypersurface $\Pi \subset \mathbb{P}^5$ contains two systems of 2-dimensional linear subspaces. A plane of the first system is defined by a point $\xi \in \mathbb{P}^3$ and consists of all points of Π corresponding to lines $l \subset \mathbb{P}^3$ through ξ. A plane of the second system is defined by a plane $\Xi \subset \mathbb{P}^3$ and consists of all points of Π corresponding to lines $l \subset \mathbb{P}^3$ contained in Ξ. There are no other planes contained in Π.

12 Let $F(x_0, x_1, x_2, x_3)$ be an arbitrary form of degree 4. Prove that there exists a polynomial Φ in the coefficients of F such that $\Phi(F) = 0$ is a necessary and sufficient condition for the surface $F = 0$ to contain a line.

13 Let $Q \subset \mathbb{P}^3$ be an irreducible quadric surface and $\Lambda_X \subset \Pi$ the set of points on the Plücker hypersurface $\Pi \subset \mathbb{P}^5$ corresponding to lines contained in Q. Prove that Λ_X consists of two disjoint conics.