Any invalid argument of this general type can be proved invalid by describing a possible non-empty universe for which its equivalent truth-functional argument is proved invalid by the method of assigning truth values.

EXERCISES

I. Prove that each of the following arguments is invalid:

1. All ballerinas are feminine. Annette is not a ballerina. Therefore Annette is not feminine.
2. All animals are visible. All unicorns are animals. Therefore some unicorns are visible.
3. Some liberals are Republicans. Some Republicans are isolationists. Therefore some isolationists are liberals.
4. All men and only men are rational. Some men are selfish. All men are animals. Some animals are not rational. Therefore there is at least one animal which is not selfish.
5. Lions and tigers are carnivorous mammals. Some lions are dangerous. All tigers have stripes. Some tigers are ferocious. No lions have stripes. Some mammals are neither ferocious nor dangerous. Therefore some carnivores are neither lions nor tigers.

II. Prove the validity or the invalidity of each of the following arguments:

1. No farmer is sophisticated. Adams is sophisticated. Therefore Adams is not a farmer.
2. No foreman is stupid. Brown is not a foreman. Therefore Brown is stupid.
3. All judges are lawyers. Some lawyers are shysters. Therefore some judges are shysters.
4. Some jailers are liberals. All liberals are shrewd. Therefore some jailers are shrewd.
5. All men who have ambition and intelligence are successful. Some ambitious men are not successful. Some intelligent men are not successful. Therefore some men have neither ambition nor intelligence.
6. All mattresses are either soft or uncomfortable. No soft mattress is uncomfortable. Some mattresses are uncomfortable. Therefore some mattresses are not soft.

7. Offices and houses are uncomfortable and disagreeable if they are either stuffy or chilly. Therefore any office will be uncomfortable if it is stuffy.
8. All men and women will be healthy and vigorous if they exercise and do not dissipate. Therefore any man will be healthy if he exercises.
9. No applicant will be either hired or considered who is either untrained or inexperienced. Some applicants are inexperienced beginners. All applicants who are women will be disappointed except those who are hired. Every applicant is a woman. Some women will be hired. Therefore some applicants will not be disappointed.
10. No candidate will be either elected or appointed who is either a liberal or a radical. Some candidates are wealthy liberals. All candidates who are politicians will be disappointed except those who are elected. Every candidate is a politician. Some politicians will be elected. Therefore some candidates will be disappointed.

IV. MULTIPLY GENERAL PROPOSITIONS

Thus far we have limited our attention to singular propositions and to general propositions which contain only a single quantifier. A general proposition which contains exactly one quantifier is said to be *singly* general. We turn next to *multiply* general propositions, which contain two or more quantifiers. In our usage of the term, any compound statement whose components are general propositions is to be counted as a multiply general proposition. For example, the conditional 'If all dogs are carnivorous then some animals are carnivorous', symbolized as '$(x)[Dx \supset Cx] \supset (\exists x)[Ax \cdot Cx]$', is a multiply general proposition. Other multiply general propositions are more complex and require a more complicated notation. To develop the new notation we must turn again to the notion of a propositional function.

All propositional functions considered up to now have had as substitution instances either singular propositions or truth-functional compounds of singular propositions having the same subject terms. But if we now consider a compound statement whose components are singular propositions having *different* subject

Similarly the proposition 'There are some *H*'s' may be alternatively symbolized as '$(\exists u)Hu$', '$(\exists v)Hv$', . . . , '$(\exists y)Hy$', or '$(\exists z)Hz$'. The difference between '$(x)Fx$' and '$(y)Fy$' (as between '$(\exists x)Gx$' and '$(\exists y)Gy$') is purely notational, and either may be written in place of the other wherever it occurs. Of course where we have a propositional function containing free occurrences of two or more different variables, such as '$Fx \cdot Gy$', the two propositional functions which result from quantifying it differently as

$$(x)[Fx \cdot Gy] \quad \text{and} \quad (y)[Fx \cdot Gy]$$

are very different indeed, and their difference is more than merely notational. The substitution instances of the first are

$$(x)[Fx \cdot Ga],\ (x)[Fx \cdot Gb],\ (x)[Fx \cdot Gc],\ \ldots$$

while the substitution instances of the second are

$$(y)[Fa \cdot Gy],\ (y)[Fb \cdot Gy],\ (y)[Fc \cdot Gy],\ \ldots$$

If it so happens that every individual has the property *F* but only some individuals have the property *G*, then some substitution instances of the first will be true propositions, while all substitution instances of the second will be false, a considerable difference indeed! This example should serve to indicate the need for speaking not of "*the* universal (or existential) quantification of a propositional function" but rather of "the universal (or existential) quantification of a propositional function *with respect to the variable* '*x*' " or "the universal (or existential) quantification of a propositional function *with respect to the variable* '*y*' " and so on.

It should be clear that since '$(x)[Fx \supset Gx]$' and '$(y)[Fy \supset Gy]$' are alternative translations of the proposition 'Everything which is an *F* is also a *G*', the universal quantification of '$Fx \supset Gx$' with respect to '*x*' has the same meaning and is logically equivalent to the universal quantification with respect to '*y*' of the propositional function which results from replacing all free occurrences of '*x*' in '$Fx \supset Gx$' by '*y*'—for the result of that replacement is '$Fy \supset Gy$'. In the early stages of our work it will be desirable to have at most one quantification with respect to a given variable in a single proposition. This is not strictly *necessary*, but it is helpful in preventing confusion. Thus the first multiply general proposition considered, 'If all dogs are carnivorous then some animals are carnivorous', is more conveniently symbolized as '$(x)[Dx \supset Cx] \supset (\exists y)[Ay \cdot Cy]$' than as '$(x)[Dx \supset Cx] \supset (\exists x)[Ax \cdot Cx]$', although neither is *incorrect*.

It has been remarked that no proposition can contain a free occurrence of any variable. Hence in symbolizing any proposition we must take care that every occurrence of every variable used lies within the scope of a quantifier with respect to that variable. Some examples will help to make the matter clear. The proposition

> If something is wrong with the house then everyone in the house complains.

is properly symbolized as a conditional whose antecedent and consequent contain different quantifiers:

$$(\exists x)[x \text{ is wrong with the house}] \supset (y)[(y \text{ is a person in the house}) \supset (y \text{ complains})].$$

Here the scope of the initial quantifier does not extend past the main implication sign. But if we turn now to another proposition which bears a superficial resemblance to the first:

> If something is wrong then it should be rectified.

it would be *incorrect* to symbolize it as

$$(\exists x)[x \text{ is wrong}] \supset (x \text{ should be rectified}).$$

For since the scope of the initial quantifier ends at the implication sign, the occurrence of '*x*' in the consequent *cannot* refer back to the initial quantifier because it does not lie within its scope. We have here a free occurrence of a variable, which means that the proposed symbolization is not a proposition and therefore not an adequate translation of the given statement. The error is not to be corrected by *simply* extending the scope terms, such as '$Fa \cdot Gb$', we can regard it as a substitution instance either of the propositional function '$Fx \cdot Gb$' or of the propositional function '$Fa \cdot Gx$'. Some propositional functions, we see, may contain singular propositions as parts. And if we consider a compound statement of which one constituent is a general proposition and the other constituent is a singular proposition, such as 'If all dogs are carnivorous then Rover is carnivorous', symbolized as '$(x)[Dx \supset Cx] \supset Cr$', we can regard it as a substitution instance of the propositional function '$(x)[Dx \supset Cx] \supset Cx$'. Thus we see that some propositional functions may contain general propositions as parts.

At this point two new technical terms may properly be introduced. An occurrence of the variable 'x' which does not occur within, or lie within the scope of, a universal or existential quantifier* '(x)' or '$(\exists x)$' will be called a *free occurrence* of that variable. On the other hand, an occurrence of the variable 'x' which is either part of a quantifier or lies within the scope of a quantifier '(x)' or '$(\exists x)$' will be called a *bound occurrence* of that variable.† Thus in the expression '$(x)[Dx \supset Cx] \supset Cx$' the first occurrence of the variable 'x' is *part of a quantifier* and is therefore considered to be *bound*. The second and third occurrences are bound occurrences also. But the fourth occurrence is a free occurrence. Thus we see that propositional functions may contain both free and bound occurrences of variables. On the other hand, all occurrences of variables in propositions must be bound, since every proposition must be either true or false. A propositional function must contain at least one free occurrence of a variable, but no proposition can contain any free occurrences of any variable.

The proposition '$Fa \cdot Gb$' can also be regarded as a substitution instance of '$Fx \cdot Gy$', where the latter is a propositional function containing *two different variables*. Up to now we have explicitly admitted only one individual variable, the letter 'x'. However, in our previous *use* of the letter 'y' to denote *any arbitrarily selected*

* As explained on page 71.

† An alternative, less common nomenclature refers to free variables as 'real' variables, and to bound variables as 'apparent' variables.

individual, we were in effect using it as a variable without admitting the fact. And in introducing the letter 'w' by **EI** to denote *some particular* individual having a specified property, without really knowing *which* individual it denoted, we were in effect using 'w' as a variable also. We now proceed to acknowledge candidly what was implicit in our former usage. Some propositional functions may contain two or more different individual variables. It will be convenient to have a larger supply of individual variables available, so we readjust our notational conventions to include the letters 'u', 'v', 'w', 'x', 'y', and 'z' as individual variables. Propositional functions now include such expressions as 'Fu', '$Fu \vee Gw$', '$(Fx \cdot Gy) \supset Hz$', '$Fx \vee (Gy \cdot Hx)$', and the like.

It should be observed that in replacing variables by constants to obtain propositions from propositional functions, the same constant must replace every free occurrence of the same variable. Thus substitution instances of the propositional function '$Fx \vee (Gy \cdot Hx)$' are

$$Fa \vee (Gb \cdot Ha),\ Fa \vee (Gc \cdot Ha),\ Fa \vee (Gd \cdot Ha),\ \ldots$$
$$Fb \vee (Ga \cdot Hb),\ Fb \vee (Gc \cdot Hb),\ Fb \vee (Gd \cdot Hb),\ \ldots$$
$$Fc \vee (Ga \cdot Hc),\ Fc \vee (Gb \cdot Hc),\ Fc \vee (Gd \cdot Hc),\ \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

but *not* such propositions as '$Fa \vee (Gb \cdot Hc)$'. On the other hand, the *same* constant can replace free occurrences of *different* variables, provided, of course, that if it replaces any free occurrence of a variable it must replace all free occurrences of that variable. Thus additional substitution instances of the propositional function '$Fx \vee (Gy \cdot Hx)$' are '$Fa \vee (Ga \cdot Ha)$', '$Fb \vee (Gb \cdot Hb)$', '$Fc \vee (Gc \cdot Hc)$', . . .

Having admitted the letters 'u', 'v', 'w', 'y', and 'z' as individual variables in addition to 'x', we now adjust our notation for universal and existential quantification to conform to our expanded stock of variables. The proposition 'All F's are G's' may be alternatively symbolized as '$(u)[Fu \supset Gu]$', '$(v)[Fv \supset Gv]$', '$(w)[Fw \supset Gw]$', '$(x)[Fx \supset Gx]$', '$(y)[Fy \supset Gy]$', or '$(z)[Fz \supset Gz]$'.

of the initial quantifier through rebracketing, moreover, for the symbolic expression

$$(\exists x)[(x \text{ is wrong}) \supset (x \text{ should be rectified})]$$

although a proposition, does not have the same meaning as the original proposition in English. Instead, it says merely that there is at least one thing which should be rectified if it is wrong, but the sense of the English sentence is clearly that if *anything* is wrong then it should be rectified. Hence a correct symbolization is neither of the preceding, but rather

$$(x)[(x \text{ is wrong}) \supset (x \text{ should be rectified})].$$

The situation is more complicated, but no different in principle, when one quantifier occurs *within the scope of another quantifier*. Here the same warning against dangling or unquantified variables must be sounded. The proposition

> If something is missing then if nobody calls the police someone will be unhappy.

is properly symbolized as

$$(\exists x)[x \text{ is missing}] \supset \{(y)[(y \text{ is a person}) \supset \sim(y \text{ calls the police})] \supset (\exists z)[(z \text{ is a person})\cdot(z \text{ will be unhappy})]\}.$$

But the following proposition, which is superficially analogous to the preceding:

> If something is missing then if nobody calls the police it will not be recovered.

is *not* to be symbolized as

$$(\exists x)[x \text{ is missing}] \supset \{(y)[(y \text{ is a person}) \supset \sim(y \text{ calls the police})] \supset \sim(x \text{ will be recovered})\}$$

for the last occurrence of the variable '*x*' is outside the scope of the initial quantifier, being left dangling. It too cannot be corrected simply by rebracketing, as

$$(\exists x)\{(x \text{ is missing}) \supset \{(y)[(y \text{ is a person}) \supset \sim(y \text{ calls the police})] \supset \sim(x \text{ will be recovered})\}\}$$

for this expression fails equally to preserve the sense of the English sentence, in the same way as in the previous example. That sense is expressed by the formula

$$(x)\{(x \text{ is missing}) \supset \{(y)[(y \text{ is a person}) \supset \sim(y \text{ calls the police})] \supset \sim(x \text{ will be recovered})\}\}$$

which is therefore a correct symbolization of the given proposition.

EXERCISES

Symbolize each of the following propositions, in each case using the abbreviations which are suggested:

1. If anything is missing someone will call the police. (*Mx*-*x* is missing, *Px*-*x* is a person, *Cx*-*x* will call the police.)
2. If anything is missing the maid probably took it. (*Mx*-*x* is missing, *Tx*-*x* was probably taken by the maid.)
3. If any diamonds are large then some diamonds are expensive. (*Dx*-*x* is a diamond, *Lx*-*x* is large, *Ex*-*x* is expensive.)
4. If any diamonds are large then, if all large diamonds are expensive, they are expensive. (*Dx*-*x* is a diamond, *Lx*-*x* is large, *Ex*-*x* is expensive.)
5. If all students who are present are either botany majors or zoology majors then either some botany majors are present or some zoology majors are present. (*Sx*-*x* is a student, *Px*-*x* is present, *Bx*-*x* is a botany major, *Zx*-*x* is a zoology major.)
6. If any student is present then either no botany majors are present or he is a botany major. (*Sx*-*x* is a student, *Px*-*x* is present, *Bx*-*x* is a botany major.)
7. If all visitors are friendly and only relatives are visitors then if there are any visitors some relatives are friendly. (*Vx*-*x* is a visitor, *Fx*-*x* is friendly, *Rx*-*x* is a relative.)
8. If there are any visitors and only relatives are visitors then they must be relatives. (*Vx*-*x* is a visitor, *Rx*-*x* is a relative.)
9. If all wives are ambitious and no husbands are successful then some wives will be miserable. (*Wx*-*x* is a wife, *Ax*-*x* is ambitious, *Hx*-*x* is a husband, *Sx*-*x* is successful, *Mx*-*x* will be miserable.)

10. If any husband is unsuccessful then if all wives are ambitious he will be miserable. (*Hx*-*x* is a husband, *Sx*-*x* is successful, *Wx*-*x* is a wife, *Ax*-*x* is ambitious, *Mx*-*x* will be miserable.)

V. REVISED QUANTIFICATION RULES

1. Inferences Involving Propositional Functions. In constructing a formal proof or demonstration that a given argument is valid, the premisses with which we begin and the conclusion with which we end are all propositions. But wherever the rules of Existential Instantiation or Universal Generalization are used some of the intermediate steps must contain free variables, and will therefore be propositional functions rather than propositions. Each step of a demonstration must be either a premiss, or an assumption of limited scope, or follow validly from preceding steps by an elementary valid argument form, or follow from a sequence of preceding steps by the principle of Conditional Proof. Three questions naturally arise at this point: In what sense can a propositional function be said to follow *validly* from other propositional functions? In what sense can a propositional function be said to follow *validly* from propositions? And in what sense can a proposition be said to follow *validly* from propositional functions?

These questions are not difficult to answer. Propositional functions contain free variables, and are therefore neither true nor false. But a propositional function becomes a proposition when all its free variables are replaced by constants, and the resulting substitution instance is either true or false. One propositional function can be said to follow *validly* as conclusion from one or more other propositional functions as premisses when every replacement of free occurrences of variables by constants (the same constants replacing the same variables in both premisses and conclusion, of course) results in a valid argument. For example, the propositional function '*Gx*' follows validly from the propositional functions '$Fx \supset Gx$' and '*Fx*', because every replacement of '*x*' by a constant results in an argument of the form *modus ponens*. We may say of such an inference that it is

valid by the principle of *modus ponens* despite the fact that propositional functions rather than propositions are involved. It should be clear that any inference is valid which proceeds by way of any of the elementary valid argument forms of our original list, regardless of whether the premisses and conclusion are propositions or propositional functions. In passing we may note that this is so even where the conclusion contains more free variables than the premisses, as when by the principle of Addition we validly infer the propositional function of two variables '$Fx \vee Gy$' from the propositional function of one variable '*Fx*'.

The original list of elementary valid argument forms also permits the inference of propositional functions from propositions, as when by the principle of Addition we infer the propositional function '$Fa \vee Gx$' from the proposition '*Fa*'. That such inferences as these are valid, in the sense explained, is obvious. Moreover, propositions can validly be inferred from propositional functions by elementary valid arguments, as when by the principle of Simplification we infer the proposition '*Fa*' from the propositional function '$Fa \cdot Gx$'.

We can now adopt a more general definition of formal proof or demonstration, which parallels our earlier definition exactly except that steps of a proof can be either propositions or propositional functions. If each step which follows the initial premisses follows validly from preceding steps, in the generalized sense of 'valid' just explained, then the last step validly follows from the initial premisses. And if our initial premisses and conclusion are propositions rather than propositional functions, then the conclusion validly follows from the initial premisses in the original sense of valid which applies to arguments whose premisses and conclusions are all statements or propositions. This fact can be seen by the following considerations. As we go from our original premisses to propositional functions, if we go validly, then if the premisses are true, all substitution instances of the inferred propositional functions must be true also. And as we go from previously inferred propositional functions to other propositional functions, if we go validly, then all substitution instances of the missing then if at least one servant is honest it will be returned (Jx-x is jewelry, Mx-x is missing, Sx-x is a servant, Hx-x is honest, Rx-x will be returned.)

9. If there are any liberals then all philosophers are liberals. If there are any humanitarians, then all liberals are humanitarians. So if there are any humanitarians who are liberals then all philosophers are humanitarians. (Lx-x is a liberal, Px-x is a philosopher, Hx-x is a humanitarian.)
10. If something is lost then if everyone values his possessions it will be missed. If anyone values his possessions, so does everyone. Therefore if something is lost then if someone values his possessions then something will be missed. (Lx-x is lost, Px-x is a person, Vx-x values his possessions, Mx-x is missed.)

VI. LOGICAL TRUTHS INVOLVING QUANTIFIERS

In Chapter Two truth tables were used not only to establish the *validity* of certain arguments but also to certify the *logical truth* of certain propositions (tautologies such as '$A \vee \sim A$'). The notion of a logically true proposition is therefore a familiar one. As we have seen, not all valid arguments can be established by the method of truth tables: some of them must be demonstrated by means of our quantification rules. Similarly, not all logically true propositions can be certified by the method of truth tables: some of them must be *demonstrated* by means of our quantification rules.

The method used in demonstrating the logical truth of tautologies was set forth in Chapter Three. A demonstration of the logical truth of the tautology '$A \supset (A \vee B)$' can be set down as

→1. A	
2. $A \vee B$	1, Add.
3. $A \supset (A \vee B)$	1–2, C.P.

In demonstrating the logical truth of propositions involving quantifiers, we shall have to appeal not only to the original list of elementary valid argument forms and the strengthened principle of Conditional Proof, but to our four quantification rules

as well. Thus a demonstration of the logical truth of the proposition '$(x)Fx \supset (\exists x)Fx$' can be set down as

→1. $(x)Fx$	
2. Fy	1, **UI**
3. $(\exists x)Fx$	2, **EG**
4. $(x)Fx \supset (\exists x)Fx$	1–3, C.P.

(In discussing 'logical truth' we explicitly limit our consideration to non-empty universes, just as in discussing *validity*.)

Other logically true propositions involving quantifiers require more complicated demonstrations. For example, the logically true proposition '$(x)Fx \supset \sim(\exists x) \sim Fx$' has the following demonstration:

→1. $(x)Fx$	
→2. $(\exists x) \sim Fx$	
3. $\sim Fy$	2, **EI**
4. $(\exists x) \sim Fx \supset \sim Fy$	2–3, C.P.
5. $Fy \supset \sim(\exists x) \sim Fx$	4, Trans., D.N.
6. Fy	1, **UI**
7. $\sim(\exists x) \sim Fx$	5, 6, M.P.
8. $(x)Fx \supset \sim(\exists x) \sim Fx$	1–7, C.P.

Similarly, the truth of '$\sim(\exists x) \sim Fx \supset (x)Fx$' is demonstrated by the following:

→1. $\sim(\exists x) \sim Fx$	
→2. $\sim Fy$	
3. $(\exists x) \sim Fx$	2, **EG**
4. $(\exists x) \sim Fx \vee Fy$	3, Add.
5. Fy	4, 1, D.S.
6. $\sim Fy \supset Fy$	2–5, C.P.
7. $Fy \vee Fy$	6, Impl., D.N.
8. Fy	7, Taut.
9. $(x)Fx$	8, **UG**
10. $\sim(\exists x) \sim Fx \supset (x)Fx$	1–9, C.P.

Given the logical truths established by the two preceding demonstrations, we can conjoin them to obtain the equivalence '$(x)Fx \equiv \sim(\exists x) \sim Fx$', which was already noted as a logical truth in Section I of the present chapter. Since our proof of this equivalence did not depend upon any peculiarities of the propositional function 'Fx', the equivalence holds for *any* propositional function. And since our proof did not refer to any peculiarities of the variable 'x', the equivalence holds not only for any propositional function but for any individual variable. The equivalence *form* $(\nu)\Phi\nu \equiv \sim(\exists\nu) \sim\Phi\nu$ is thus seen to be *logically true*, and can be added to the other logical equivalences in our list of elementary valid argument forms. It permits us validly to interchange $(\nu)\Phi\nu$ and $\sim(\exists\nu) \sim\Phi\nu$ wherever they may occur. This connection between the two quantifiers by way of negation will now be adopted as an additional rule of inference, and may be used in constructing subsequent demonstrations. When it is so used, the letters '**QN**' (for *quantifier negation*) should be written to indicate which principle is being appealed to. It should be obvious that the forms

$$\sim(\nu)\Phi\nu \equiv (\exists\nu) \sim\Phi\nu$$
$$(\nu) \sim\Phi\nu \equiv \sim(\exists\nu)\Phi\nu$$
$$\sim(\nu) \sim\Phi\nu \equiv (\exists\nu)\Phi\nu$$

are all logically equivalent to each other and to the form **QN**, and are therefore logically true.

Some fairly obvious logical truths are simply stated and easily proved with our present symbolic apparatus. A logically true equivalence, for any propositional functions 'Fx' and 'Gx', is

$$[(x)Fx \cdot (x)Gx] \equiv (x)(Fx \cdot Gx)$$

which asserts that: everything has the property F and everything has the property G if and only if everything has the properties F and G. The demonstrations of the two implications involved may be written side by side:

→1. $(x)Fx \cdot (x)Gx$	
2. $(x)Fx$	1, Simp.
3. $(x)Gx$	1, Simp.
4. Fy	2, **UI**
5. Gy	3, **UI**
6. $Fy \cdot Gy$	4, 5, Conj.
7. $(x)(Fx \cdot Gx)$	6, **UG**
8. $[(x)Fx \cdot (x)Gx] \supset (x)(Fx \cdot Gx)$	1–7, C.P.

→1. $(x)(Fx \cdot Gx)$	
2. $Fy \cdot Gy$	1, **UI**
3. Fy	2, Simp.
4. Gy	2, Simp.
5. $(x)Fx$	3, **UG**
6. $(x)Gx$	4, **UG**
7. $(x)Fx \cdot (x)Gx$	5, 6, Conj.
8. $(x)(Fx \cdot Gx) \supset [(x)Fx \cdot (x)Gx]$	1–7 C.P.

Another logical equivalence involves the disjunction of existential quantifications:

$$[(\exists x)Fx \vee (\exists x)Gx] \equiv (\exists x)(Fx \vee Gx)$$

It states that if either something has the property F or something has the property G, then something has either the property F or the property G, and conversely. It may be demonstrated as follows: First we prove $[(\exists x)Fx \vee (\exists x)Gx] \supset (\exists x)(Fx \vee Gx)$:

→1. $(\exists x)Fx \vee (\exists x)Gx$	
→2. $(\exists x)Fx$	
3. Fy	2, **EI**
4. $Fy \vee Gy$	3, Add.
5. $(\exists x)(Fx \vee Gx)$	4, **EG**
6. $(\exists x)Fx \supset (\exists x)(Fx \vee Gx)$	2–5, C.P.
→7. $(\exists x)Gx$	
8. Gz	7, **EI**
9. $Fz \vee Gz$	8, Add.
10. $(\exists x)(Fx \vee Gx)$	9, **EG**
11. $(\exists x)Gx \supset (\exists x)(Fx \vee Gx)$	7–10, C.P.
12. $[(\exists x)Fx \supset (\exists x)(Fx \vee Gx)] \cdot [(\exists x)Gx \supset (\exists x)(Fx \vee Gx)]$	6, 11, Conj.
13. $(\exists x)(Fx \vee Gx) \vee (\exists x)(Fx \vee Gx)$	12, 1, C.D.
14. $(\exists x)(Fx \vee Gx)$	13, Taut.
15. $[(\exists x)Fx \vee (\exists x)Gx] \supset (\exists x)(Fx \vee Gx)$	1–14, C.P.

Then we prove $(\exists x)(Fx \vee Gx) \supset [(\exists x)Fx \vee (\exists x)Gx]$:

1. $(\exists x)(Fx \vee Gx)$	
2. $Fy \vee Gy$	1, **EI**
3. Fy	
4. $(\exists x)Fx$	3, **EG**
5. $(\exists x)Fx \vee (\exists x)Gx$	4, Add.
6. $Fy \supset [(\exists x)Fx \vee (\exists x)Gx]$	3–5, C.P.
7. Gy	
8. $(\exists x)Gx$	7, **EG**
9. $(\exists x)Fx \vee (\exists x)Gx$	8, Add.
10. $Gy \supset [(\exists x)Fx \vee (\exists x)Gx]$	7–9, C.P.
11. $\{Fy \supset [(\exists x)Fx \vee (\exists x)Gx]\} \cdot \{Gy \supset [(\exists x)Fx \vee (\exists x)Gx]\}$	6, 10, Conj.
12. $[(\exists x)Fx \vee (\exists x)Gx] \vee [(\exists x)Fx \vee (\exists x)Gx]$	11, 2, C.D.
13. $(\exists x)Fx \vee (\exists x)Gx$	12, Taut.
14. $(\exists x)(Fx \vee Gx) \supset [(\exists x)Fx \vee (\exists x)Gx]$	1–13, C.P.

Another logical truth is in the form of a conditional rather than an equivalence. Written as

$$[(x)Fx \vee (x)Gx] \supset (x)(Fx \vee Gx)$$

it asserts that if either everything is an F or everything is a G, then everything is either an F or a G. Its demonstration too involves making several assumptions of limited scope, and can be written as follows:

1. $(x)Fx \vee (x)Gx$	
2. $(x)Fx$	
3. Fy	2, **UI**
4. $Fy \vee Gy$	3, Add.
5. $(x)(Fx \vee Gx)$	4, **UG**
6. $(x)Fx \supset (x)(Fx \vee Gx)$	2–5, C.P.
7. $(x)Gx$	
8. Gy	7, **UI**
9. $Fy \vee Gy$	8, Add.
10. $(x)(Fx \vee Gx)$	9, **UG**
11. $(x)Gx \supset (x)(Fx \vee Gx)$	7–10, C.P.
12. $[(x)Fx \supset (x)(Fx \vee Gx)] \cdot [(x)Gx \supset (x)(Fx \vee Gx)]$	6, 11, Conj.
13. $(x)(Fx \vee Gx) \vee (x)(Fx \vee Gx)$	12, 1, C.D.
14. $(x)(Fx \vee Gx)$	13, Taut.
15. $[(x)Fx \vee (x)Gx] \supset (x)(Fx \vee Gx)$	1–14, C.P.

The converse of this conditional is *not* logically true, however. The converse states that if everything is either F or G then either everything is F or everything is G. That this converse is not always true can be seen by replacing 'G' by '$\sim F$', for '$(x)(Fx \vee \sim Fx)$' is true for any predicate 'F', while there are few for which '$(x)Fx \vee (x) \sim Fx$' holds. Another logical truth which is conditional in form is

$$(\exists x)(Fx \cdot Gx) \supset [(\exists x)Fx \cdot (\exists x)Gx].$$

Its demonstration is perfectly straightforward, and can be left as an exercise for the reader. That its converse is not true in general can be seen by again replacing 'G' by '$\sim F$'. For most predicates 'F' the proposition '$(\exists x)Fx \cdot (\exists x) \sim Fx$' is true (e.g. 'something is round and something is not round'), but for any 'F' the proposition '$(\exists x)(Fx \cdot \sim Fx)$' is logically false.

It has already been observed that propositional functions can contain propositions as constituent parts. Examples of such propositional functions are

$$Fx \cdot Ga,\ Gy \vee (z)Hz,\ (\exists w)Gw \supset Fz,\ \ldots$$

When such propositional functions as these are quantified, to obtain the propositions

$$(x)[Fx \cdot Ga],\ (\exists y)[Gy \vee (z)Hz],\ (z)[(\exists w)Gw \supset Fz],\ \ldots$$

we have propositions lying within the scopes of quantifiers, although the quantifiers have no real affect on those propositions. When a quantifier with respect to a given variable is prefixed to an expression its only effect is to bind previously free occurrences of that variable. In the expressions written above, the propositions 'Ga', '$(z)Hz$', and '$(\exists w)Gw$', although lying within the scopes of the quantifiers '(x)', '$(\exists y)$', and '(z)', respectively, are not really affected by them. Wherever we have an expression containing a quantifier within whose scope a *proposition* lies, the entire expression is logically equivalent to another expression in which the scope of the quantifier does *not* extend over the proposition in question. An example or two will make this clear.

In the following, let 'p' be any *proposition*, and 'Fx' any propositional function containing at least one free occurrence of the variable 'x'. Our first equivalence here is between the universal quantification of '$Fx \cdot p$' and the conjunction of the universal quantification of 'Fx' with 'p', which is more briefly expressed as

$$(x)(Fx \cdot p) \equiv [(x)Fx \cdot p].$$

The demonstration of this equivalence can be written as

→1. $(x)(Fx \cdot p)$	
2. $Fy \cdot p$	1, UI
3. Fy	2, Simp.
4. $(x)Fx$	3, UG
5. p	2, Simp.
6. $(x)Fx \cdot p$	4, 5, Conj.
7. $(x)(Fx \cdot p) \supset [(x)Fx \cdot p]$	1–6, C.P.

→1. $(x)Fx \cdot p$	
2. $(x)Fx$	1, Simp.
3. Fy	2, UI
4. p	1, Simp.
5. $Fy \cdot p$	3, 4, Conj.
6. $(x)(Fx \cdot p)$	5, UG
7. $[(x)Fx \cdot p] \supset (x)(Fx \cdot p)$	1–6, C.P.

Another logical equivalence holds between the universal quantification of '$p \supset Fx$' and the conditional statement whose antecedent is 'p' and whose consequent is the universal quantification of 'Fx'. The first asserts that *given any individual x, p implies that x has F*, and is equivalent to *p implies that given any individual x, x has F*. Our symbolic expression of this equivalence is

$$(x)(p \supset Fx) \equiv [p \supset (x)Fx].$$

Its demonstration is easily constructed:

→1. $(x)(p \supset Fx)$	
2. $p \supset Fy$	1, UI
→3. p	
4. Fy	2, 3, M.P.
5. $(x)Fx$	4, UG
6. $p \supset (x)Fx$	3–5, C.P.
7. $(x)(p \supset Fx) \supset [p \supset (x)Fx]$	1–6, C.P.

→1. $p \supset (x)Fx$	
→2. p	
3. $(x)Fx$	1, 2, M.P.
4. Fy	3, UI
5. $p \supset Fy$	2–4, C.P.
6. $(x)(p \supset Fx)$	5, UG
7. $[p \supset (x)Fx] \supset (x)(p \supset Fx)$	1–6, C.P.

The same pattern of equivalence holds for the existential quantification of '$p \supset Fx$' and the conditional statement '$p \supset (\exists x)Fx$'.

The first asserts that *there is at least one individual x such that p implies that x has F*, and is equivalent to *p implies that there is at least one individual x such that x has F*, which is asserted by the second. Its demonstration is very easily constructed, and will be left as an exercise.

However, the pattern of equivalence is different when 'p' occurs as consequent rather than antecedent. Although the universal quantification of '$Fx \supset p$' implies '$(x)Fx \supset p$', it is not implied by the latter. There is an equivalence, however, between *given any x, if x has F then p* and *if there is at least one x such that x has F, then p*, which is expressed symbolically as

$$(x)(Fx \supset p) \equiv [(\exists x)Fx \supset p].$$

And although the existential quantification of '$Fx \supset p$' is implied by '$(\exists x)Fx \supset p$', it does not imply the latter. There is an equivalence, however, between *there is at least one x such that if x has F then p* and *if given any x, x has F, then p*, which is expressed symbolically as

$$(\exists x)(Fx \supset p) \equiv [(x)Fx \supset p].$$

The first is demonstrated as follows:

→1. $(x)(Fx \supset p)$	
→2. $(\exists x)Fx$	
3. Fy	2, EI
4. $Fy \supset p$	1, UI
5. p	4, 3, M.P.
6. $(\exists x)Fx \supset p$	2–5, C.P.
7. $(x)(Fx \supset p) \supset [(\exists x)Fx \supset p]$	1–6, C.P.

→1. $(\exists x)Fx \supset p$	
→2. Fy	
3. $(\exists x)Fx$	2, EG
4. $Fy \supset (\exists x)Fx$	2–3, C.P.
5. $Fy \supset p$	4, 1, H.S.
6. $(x)(Fx \supset p)$	5, UG
7. $[(\exists x)Fx \supset p] \supset (x)(Fx \supset p)$	1–6, C.P.

The present logical equivalence supplies an alternative method of symbolizing one of the propositions discussed in Section IV:

> If something is wrong with the house then everyone in the house complains.

The translation given there abbreviates to

$$(\exists x)(Wx) \supset (y)(Py \supset Cy)$$

which, as we have just demonstrated, is logically equivalent to

$$(x)[Wx \supset (y)(Py \supset Cy)].$$

We shall conclude our discussion of logically true propositions involving quantifiers by turning our attention to four logically true propositions which are neither equivalences nor conditionals. They correspond, in a sense, to our four quantification rules:

1. $(y)[(x)Fx \supset Fy]$
2. $(y)[Fy \supset (\exists x)Fx]$
3. $(\exists y)[Fy \supset (x)Fx]$
4. $(\exists y)[(\exists x)Fx \supset Fy]$

The first of these corresponds to **UI**, saying, as it does, that given any individual y, if every individual has the property F, then y does. Its demonstration is almost trivially obvious, proceeding:

→1. $(x)Fx$	
2. Fz	1, **UI**
3. $(x)Fx \supset Fz$	1–2, C.P.
4. $(y)[(x)Fx \supset Fy]$	3, **UG**

The second corresponds to **EG**, asserting that if any given individual y has the property F, then something has F. The third and fourth, corresponding to **UG** and **EI**, respectively, are not so immediately obvious, but nevertheless are logically true and quite easily demonstrated. An intuitive explanation can be given by reference to the ancient Athenian general and statesman Aristides, often called 'the just'. So outstanding was Aristides for his rectitude that the Athenians had a saying that

If anyone is just, Aristides is just.

With respect to *any* property, there is always some individual y such that if anything has that property, y has it. That is what is asserted by the fourth proposition listed above, which corresponds to **EI**. The matter can be put another way. If we turn our attention not to the property of being just, but to its reverse, the property of being corruptible, then the sense of the first

Athenian saying is also expressible as

If Aristides is corruptible, then everyone is corruptible.

Again generalizing, we may observe that with respect to *any* property there is always some individual y such that if y has that property, everything has it. That is what is asserted by the third proposition listed above, which corresponds to **UG**. Its demonstration proceeds:

→1. $\sim(x)Fx$	
2. $(\exists x) \sim Fx$	1, **QN**
3. $\sim Fz$	2, **EI**
4. $\sim(x)Fx \supset \sim Fz$	1–3, C.P.
5. $Fz \supset (x)Fx$	4, Trans.
6. $(\exists y)[Fy \supset (x)Fx]$	5, **EG**

Although we shall not prove it until the end of Chapter Nine, the methods of proof so far assembled (techniques for 'Natural Deduction', as they are sometimes called) permit the demonstration of all logically true propositions constructed out of truth-functional connectives and the quantification of individual variables. It will also be proved that *only* propositions that are logically true can be demonstrated by these techniques.

EXERCISES

Construct demonstrations for the following:

1. $(\exists x)(Fx \cdot Gx) \supset [(\exists x)Fx \cdot (\exists x)Gx]$
2. $(x)(Fx \supset Gx) \supset [(x)Fx \supset (x)Gx]$
3. $[(\exists x)Fx \supset (\exists x)Gx] \supset (\exists x)(Fx \supset Gx)$
4. $(\exists x)(p \supset Fx) \equiv [p \supset (\exists x)Fx]$
5. $(\exists x)(Fx \cdot p) \equiv [(\exists x)Fx \cdot p]$
6. $(x)(Fx \vee p) \equiv [(x)Fx \vee p]$
7. $(\exists x)(Fx \vee p) \equiv [(\exists x)Fx \vee p]$
8. $(\exists x)(Fx \supset p) \equiv [(x)Fx \supset p]$
9. $(y)[Fy \supset (\exists x)Fx]$
10. $(\exists y)[(\exists x)Fx \supset Fy]$