ALGEBRAIC APPROACH TO TROPICAL MATHEMATICS

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Abstract. Tropical mathematics is a new area of mathematics that utilizes degeneration to simplify combinatorial geometric problems to facilitate their solution. This degeneration can often be described in terms of valuation theory, which leads to an interesting algebraic structure that will be the focus of the course.

Tropical mathematics often is defined over an ordered cancellative monoid $\mathcal{M}$, usually taken to be $(\mathbb{R}, +)$ or $(\mathbb{Q}, +)$, which can also be viewed as the max-plus semiring, which is (additively) idempotent. Although a rich theory has arisen from this viewpoint, cf. [L1], idempotent semirings possess a restricted algebraic structure theory, and also do not reflect certain valuation-theoretic properties, thereby forcing researchers to rely often on combinatoric techniques.

In this course we describe an alternative structure, more compatible with valuation theory, studied by the authors over the past few years, that permits fuller use of algebraic theory especially in understanding the underlying tropical geometry.

The idempotent max-plus algebra A of an ordered monoid $\mathcal{M}$ is replaced by $R := L \times \mathcal{M}$, where L is a given indexing semiring (not necessarily with 0). In this case we say R **layered** by L. When L is trivial, i.e, $L = \{1\}$, R is the usual bipotent max-plus algebra. When $L = \{1, \infty\}$ we recover the "standard" supertropical structure with its "ghost" layer. When $L = \mathbb{N}$ we can describe multiple roots of polynomials via a "layering function" $s : R \to L$.

Likewise, one can define the layering $s : R^{(n)} \to L^{(n)}$ componentwise; vectors $v_1, \dots, v_m$ are called **tropically dependent** if each component of some nontrivial linear combination $\sum \alpha_i v_i$ is a ghost, for "tangible" $\alpha_i \in R$. Then an $n \times n$ matrix has tropically dependent rows iff its permanent is a ghost.

We explain how supertropical algebras, and more generally layered algebras, provide a robust algebraic foundation for tropical linear algebra, in which many classical tools are available. In the process, we provide some new results concerning the rank of d-independent sets (such as the fact that they are semi-additive), put them in the context of supertropical bilinear forms, and lay the matrix theory in the framework of identities of semirings.

1. SYLLABUS

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2. OVERVIEW

Tropical geometry, a rapidly growing area expounded for example in [Gat, ItMS, L1, MS, SS], has been based on two main approaches. The most direct passage to tropical mathematics is via logarithms. But valuation theory has richer algebraic applications (for example providing a quick proof of Kapranov's theorem), and much of tropical geometry is based on valuations on Puiseux series. The structures listed above are compatible with valuations, and in §3.4 we see how valuations fit in with this approach.

Since tropicalization as a process of degeneration into simpler concepts, one is motivated to develop the algebraic tools at the tropical level, in order to provide an intrinsic theory to support tropical geometry as well as related subjects such as tropical linear algebra. One basic algebraic structure of tropical geometry is the max-plus algebra, which is also viewed algebraically as an ordered Abelian monoid. Considerable recent activity [CHWW, W] concerns geometry over monoids, but the ordering provides extra structure which enables us to draw on classical algebraic structure theory. The max-plus algebra is fine for framing many combinatoric questions, but it turns out that a more sophisticated structure is needed to understand the algebraic structure connected with valuations.

After introducing the basic concepts, our overlying objective is to translate ordered monoids into an algebraic theory supporting tropical linear algebra and geometry, using the following approaches:

- Algebraic geometry as espoused by Zariski and Grothendieck, using varieties and commutative algebra in the context of category theory.
- Linear algebra via tropical dependence, the characteristic polynomial, and (generalized) eigenspaces.
- Algebraic formulations for more sophisticated concepts such as resultants, discriminants, and Jacobians.

This approach leads to the use of polynomials and matrices, which requires two operations. Our task has been to pinpoint the appropriate category of semirings in which to work, or equivalently, how much far do we dequantize in the process of tropicalization? In this course we compare four structures, listed in increasing level of refinement:

- The max-plus algebra,
- Supertropical algebra,
- Layered tropical algebras,
- Exploded supertropical algebras.

2.1. **Preliminaries.**

We recall a few basic structures from algebra:

Definition 2.1. *A* ***semigroup*** *is a set endowed with an associative operation. A* ***monoid*** *is a semigroup possessing a unit element 1. A monoid is Abelian if its operation is commutative. An Abelian monoid* $\mathcal{M} := (\mathcal{M}, \cdot, 1)$ *is* ***cancellative*** *if* $ab = ac$ *implies* $b = c$.

A ***poset*** $\mathcal{A} = (\mathcal{A}, \alpha)$ *is a set* $\mathcal{A}$ *with a partial ordering* $\prec$*, i.e., a relation which is transitive and reflexive but antisymmetric (*$a \prec b$ *and* $b \prec a$ *imply* $a = b$*). One example is the* **power set** $\mathcal{P}(S)$ *of a set* S*, which is the set of subsets of* S*, where* $\prec$ *is just* $\subseteq$*. Given a poset* L *we can define the* ***dual poset*** L *obtained with* $>$ *and* $<$ *interchanged. When the partial order is total, we just call it an* ***order****.*

A ***chain*** *is a poset in which the given partial order is total.*

A ***lattice*** *is a poset in which every pair* $\{a, b\}$ *of elements has a* **supremum** *(sup)* $a \vee b$ *and an* **infinum** *(inf)* $a \wedge b$*. In other words,* $a \vee b \succeq a$ *and* $a \vee b \succeq b$*, but* $a \vee b \preceq c$ *whenever* $c \succeq a$ *and* $c \succeq b$*; analogously for* $\wedge$*, where* $\preceq$ *and* $\succeq$ *are interchanged. If* L *is a lattice then the* **dual lattice** *is defined as the* ***dual poset*** *to* L *obtained with* $\vee$ *and* $\wedge$ *interchanged.*

Although we are interested mostly in chains, lattices at times provide a useful general umbrella for the theory.

An Abelian monoid is ***ordered*** *if it has a total order satisfying the property:*

$$a \leq b \quad \textit{implies} \quad ga \leq gb, \tag{2.1}$$

for all elements a, b, g.

More generally, an Abelian monoid is **partially ordered** if it has a partial order satisfying (2.1).

Any ordered cancellative Abelian monoid is infinite.

3. Degeneration, amoebas, and general background

Our objective here is to introduce the degeneration procedure which is the underpinning of tropical mathematics. Traditionally, this is done via amoebas, so we start off with these, and use them as motivation for introducing an assortment of mathematical tools for their study, each of which brings in the insights that accompany its mathematical theory.

3.1. Amoebas.

The main motivation for tropical mathematics is to degenerate a complex hypersurface from affine algebraic geometry. The basic tropicalization, or **dequantization**, involves taking logarithms to $(\mathbb{R}, +)$. In other words, we take the variety of a polynomial, and for each point we replace the coordinates by the logarithm of their absolute value. When working over the complex plane, our original hypersurface is replaced by a 2-dimensional shape that looks like an amoeba.

The classic example that appears in all expositions of tropical mathematics is the complex line

$$ax + by + c = 0$$

where the coefficients $a, b, c \in \mathbb{C}$. Replacing x by $\frac{c}{a}x$ and y by $\frac{c}{b}y$ yields $cx + cy + c = 0$, or

$$x + y + 1 = 0, \tag{3.1}$$

so we may assume that after a change of coordinates we have (3.1). Now let

$$\lambda_1 = \log_t(|x|), \qquad \lambda_2 = \log_t(|y|),$$

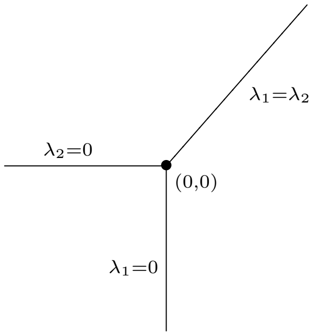
where the base t of the logarithm is to be determined. There are three asymptotics involved here:

- If $\lambda_1 \to -\infty$ then $|x| \to 0$, so $y \to 1$, and $|\lambda_2| \to 0$.
- If $\lambda_2 \to -\infty$ then $|y| \to 0$, so $x \to 1$, and $|\lambda_1| \to 0$.
- If $\lambda_1 \to \infty$ then dividing through by λ_1 yields $|\frac{\lambda_2}{\lambda_1}| \to 1$, so λ_2 approaches λ_1.

It is easy to see that our shape has three asymptotic arms, each surrounding the following ray:

- The left half of the λ_1 axis, given by $\lambda_1 < 0$ and $\lambda_2 = 0$;
- The lower half of the λ_2 axis, given by $\lambda_1 = 0$ and $\lambda_2 < 0$;
- The ray $\lambda_1 = \lambda_2$ for $\lambda_1 > 0$.

This is called the **spine** of the amoeba (of the tropical line), and the union of these three rays is called the **tropical line**, which has the following graph:



In other words, by definition, the tropical line is the spine of each amoeba. We claim that as $t \to \infty$ the amoebas "converge to the tropical line," and we shall indicate how this happens in different points of view.

More generally, any curve on the complex plane can be "tropicalized" to an amoeba by taking the logarithms of the absolute values of its components. The number of arms of the amoeba in any asymptotic direction (including multiplicity, in degenerate cases) equals the degree of the polynomial defining the curve. For example, the amoeba of a conic will have double arms in each direction.

3.2. **Introducing the max-plus semiring.** If affine complex geometry is based on the usual operations $\cdot$ and $+$ on $\mathbb{C}$, we want to see the operations induced on the logarithmic structure. So we define binary operations $\odot_t$ and $\oplus_t$ as follows:

$$t^{\lambda_1 \odot_t \lambda_2} := t_1^{\lambda} t_2^{\lambda} = t^{\lambda_1 + \lambda_2}; \qquad t^{\lambda_1 \oplus_t \lambda_2} := t_1^{\lambda} + t_2^{\lambda}. \tag{3.2}$$

Lemma 3.1. *The new operations $\odot_t$ and $\oplus_t$ on $\mathbb{R}$ are associative, and $\odot_t$ is distributive over $\oplus_t$.*

Proof. Exponentiate the usual associative and distributive properties on $\mathbb{R}$. □

Note that $\lambda_1 \odot_t \lambda_2 = \lambda_1 + \lambda_2$, and we have replaced conventional multiplication by addition.

It is a bit trickier to see what has replaced conventional addition. Let us write $\lambda_2 = \lambda_1 + c$. Then

$$t^{\lambda_1 \oplus_t \lambda_2} = t_1^{\lambda} + t_2^{\lambda} = t_1^{\lambda} + t_1^{\lambda} t^c = t_1^{\lambda}(1 + t^c). \tag{3.3}$$

For t large enough and $c > 0$, we get $1 << t^c$, so the right side approaches $t_1^{\lambda} t^c = \lambda_2$. Thus, in the limiting case $t \to \infty$, we have replaced conventional addition by the maximum. This is called the **max-plus algebra** of $(\mathbb{R}, +)$.

Remark 3.2. *Another way of thinking of this is to consider integers x and y, fix t, and write x and y to the base t. Then for t sufficiently large the leading term in the expansion of $x + y$ is the maximum of the leading terms in the expansion of x and y if they are in different positions.*

3.2.1. *Semirings without zero.* $(\mathbb{R}, \odot_t, \oplus_t)$ has a natural algebraic structure, but we do not have an additive 0 element, since in (3.3) there is no choice of c for which the right side is t_1^{λ}. Accordingly, we require part of the definition of a ring:

A **semiring**† $(R, +, \cdot, 1)$ is a set R equipped with binary operations $+$ and $\cdot$ such that:

- $(R, +)$ is an Abelian semigroup;
- $(R, \cdot\ , \mathbb{1}_R)$ is a monoid with multiplicative identity element $\mathbb{1}_R$;
- Multiplication distributes over addition.

A **domain**† (often called **semidomain0**† in the literature) is a semiring† for which its multiplicative monoid is $(R, \cdot\ , \mathbb{1}_R)$ is cancellative.

A **semifield**† is a semiring† in which every element is (multiplicatively) invertible.

A **semiring** is a semiring† with a zero element $\mathbb{0}_R$ satisfying

$$a + \mathbb{0}_R = a, \quad a \cdot \mathbb{0}_R = \mathbb{0}_R = \mathbb{0}_R \cdot a, \quad \forall a \in R.$$

Remark 3.3. *Given a semiring† R, one can obtain a semiring $\tilde{R} = R \cup \{0\}$ by adjoining the zero element 0 formally, via the following rules:*

$$0 + a = a + 0 = a, \qquad 0a = a0 = 0, \qquad \forall a \in R.$$

We prefer to work with semirings† instead of semirings since the zero element usually needs to be treated separately and often is irrelevant.

A **semifield** is a semifield† with a zero element adjoined. Note that under this definition.

Example 3.4. *(i) Any ring is a semiring. Note however that the customary field $\mathbb{Q}$ with the usual operations is not a semifield as defined here, since $\mathbb{Q} \setminus \{0\}$ is not closed under addition.*

(ii) $R = (\mathbb{R}, \odot_t, \oplus_t)$ as defined above is a semiring†, where $\mathbb{1}_R = 0$.

(iii) We have already defined the max-plus algebra of R. But we could have done this just as easily with $\mathbb{Q}$ or even $\mathbb{Z}$. The max-plus algebras $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, and $(\mathbb{R}, +)$ are semifields†, since $+$ now is the multiplication. More generally, any ordered Abelian group $(G, +)$ gives rise to a max-plus semifield† R, where the operations are written $\odot$ and $\oplus$ and defined by:

$$a \oplus b := \max\{a, b\}; \qquad a \odot b := a + b, \qquad \forall a, b \in G.$$

Again $\mathbb{1}_R = 0_G$. In this case, one can adjoin the zero element to R, denoted as $-\infty$. Associativity and distributivity (of $\odot$ over $\oplus$) hold, but NOT negation, since $a \oplus b \neq -\infty$ unless $a = b = -\infty$.

Although the circle notation is standard for the max-plus algebra in the tropical literature, we find it difficult to read when dealing with algebraic formulae. (Compare $a^4 + 7b^3 + 4c + d$ with

$$a \odot a \odot a \odot a \oplus 7 \odot b \odot b \odot b \oplus 4 \odot c \oplus d.)$$

Thus, when appealing to the abstract theory of semirings we use the usual algebraic notation of $\cdot$ (often suppressed) and $+$ respectively for multiplication and addition, and apologize in advance for any confusion this might cause.

The max-plus algebra satisfies the property that $a + b \in \{a, b\}$; we call this property **bipotence**. In particular, the max-plus algebra, viewed as a semiring†, is **idempotent** in the sense that $a + a = a$ for all a. Although idempotence pervades the tropical literature, it turns out that what is really crucial for many applications is the following fact:

Remark 3.5. *In any idempotent semiring†, if $a + b + c = a$, then $a + b = a$. (Proof: $a = a + b + c = (a + b + c) + b = a + b$.)*

Let us call such a semiring† **proper**. Note that a proper semiring cannot have additive inverses other than $\mathbb{0}$, since if $a + c = \mathbb{0}$, then $a = \mathbb{0} + a = a + c + a$, implying $a = a + c = \mathbb{0}$.

3.3. **Topological methods: The Hausdorff metric.** We consider the usual distance metric d defined on $\mathbb{R}^{(n)}$ Given two compact surfaces $A, B \subset \mathbb{R}$, we define

$$d_H(a, B) = \inf\{d(a, b) : b \in B\}; \qquad d_H(A, b) = \inf\{d(a, b) : a \in A\};$$

$$d_H(A, B) = \max\{d_H(a, B), d_H(A, b) : a \in A, b \in B\}.$$

Note that $d_H(A, B) = 0$ iff $A = B$. One can prove [ItMS, Proposition 1.2] that under this metric, as $t \to \infty$ the amoebas of a hypersurface converge to the spine, in the sense that for any compact set C there is an open set $U \supset C$ such that

$$\lim_{t \to \infty} d_H(A_{t_1} \cap U, A_{t_2} \cap U) \to 0, \quad \forall t_1, t_2 > t.$$

Thus, topological methods apply to the study of amoebas. One can extend these considerations to consider series of polynomials whose hypersurfaces converge, rather than fixing a given hypersurface. For example, we could parameterize the complex line to

$$p_t(a)z + q_t(b)w + r_t(c) = 0,$$

and again consider the limiting case as $t \to \infty$. Here p_t, q_t, and r_t could be polynomials say in $\frac{1}{t}$, so that their limit exists.

Equivalently, we could take p_t, q_t, and r_t to be polynomials in t and take their limit as $t \to 0$, which will be our perspective from now on. Note that $\log(\frac{1}{t}) = -\log t$.

This leads us to view algebraic geometry with coefficients in $\mathbb{C}[t]$.

3.4. **Puisuex series and valuations.** Continuing the idea of §3.3, we want to switch our underlying base ring of algebraic geometry from $\mathbb{C}$ to $\mathbb{C}[t]$. But $\mathbb{C}[t]$ lacks certain basic features of $\mathbb{C}$. For starters, it is not closed under limits, which was the reason we introduced the indeterminate t at the outset. For example, we may need the limit of the series

$$t, \quad t^2+t, \quad t^3+t^2+t, \quad \dots.$$

which should be $\sum_{i\geq 1} t^i$. We can correct that by passing to the ring of **formal power series** $\mathbb{C}[[t]]$, to be defined presently. But $\mathbb{C}[[t]]$ is not a field, More generally, given any ring R, we can define the ring of **Laurent series** $\mathbb{C}((t))$. But this still lacks the property, so useful in algebraic geometry, of being algebraically closed. For example, we want $\sqrt[n]{t} = t^{1/n}$, which leads us to **Puiseux series**. Let us take a moment to define these and study them; we can start with an arbitrary field K instead of $\mathbb{C}$. (In fact, the definitions work just as well when K is a ring, or even a semiring†.) Throughout, $\mathbb{N}$ denotes the natural numbers including 0.

Definition 3.6. *Suppose K is a semiring†.*

*(i) The **polynomial semiring** $K[t]$ in the single indeterminate t is defined in the usual way, as*

$$\left\{\sum_{i\in\mathbb{N}} c_i t^i : c_i \in K\right\},$$

the sum taken over finite subsets of $\mathbb{N}$, where addition is obtained by adding coefficients, and multiplication given by the familiar formula

$$\left(\sum_i c_i t^i\right)\left(\sum_j d_j t^j\right) = \sum_{u\in\mathbb{N}}\left(\sum_{i+j=u} c_i d_j\right) t^u. \tag{3.4}$$

The product is well-defined because for any given u there are only finitely many i, j such that $i+j=u$.

*(ii) The **semiring of formal power series** $K[[t]]$ is the set of formal sums*

$$\left\{\sum_{i\in\mathbb{N}} c_i t^i : c_i \in K\right\},$$

where addition still is obtained by adding coefficients, and multiplication given by the familiar formula (3.4). The product still is well-defined because, again, for any given u there are only finitely many $i, j \in \mathbb{N}$ such that $i+j=u$.

*(iii) The **semiring of formal Laurent series** $K((t))$ is*

$$\left\{p(t) := \sum_{i\in\mathbb{Z}, i\geq i_0} c_i t^i : c_i \in K, \quad c_{i_0} \neq 0 \text{ for some minimal } i_0 \text{ depending on } p\right\},$$

where again addition is obtained by adding coefficients, and multiplication $(\sum_{i\in\mathbb{Z}, i\geq i_0} c_i t^i)(\sum_{j\in\mathbb{Z}, j\geq j_0} d_j t^j)$ given by the formula (3.4), except that now we permit $u \geq i_0 + j_0$. The product still is well-defined because, again, for any given u there are only finitely many $i \geq i_0$ and $j \geq j_0$ in $\mathbb{Z}$ such that $i+j=u$.

*(iv) The **semiring of Puiseux series** $K((t))_{\mathrm{Puis};\mathcal{G}}$ is*

$$\left\{p(t) := \sum_{i\in\mathcal{G},\ c_i\in K} c_i t^i\right\},$$

where $\mathcal{G}$ is a given subgroup of $(R, +)$, and the powers of t are taken over well-ordered subsets of $\mathcal{G}$ (depending on p). Again addition is obtained by adding coefficients, and multiplication given by the formula (3.4), except that now we permit arbitrary $u \in \mathcal{G}$. Again, for any $u \in \mathcal{G}$ there are only finitely many $i, j \in \mathcal{G}$ such that $c_i \neq 0$, $c_j \neq 0$, and $i+j=u$, seen from the following argument:

Let

$$I = \{i \in \mathcal{G} : c_i \neq 0,\ c_{u-j} \neq 0\}.$$

We need to show that I is finite. Take $i_1 \in I_1$ minimal, $i_2 \in I \setminus \{i_1\}$ minimal, $i_3 \in I \setminus \{i_1, i_2\}$ minimal, and so forth. The set $\{u-i_1, u-i_2, \dots\}$ has some minimal element $u-i_m$, and then it follows that $|I| = m$.

It is customary in the literature to take $\mathcal{G} = (\mathbb{R}, +)$, but we shall see that algebraically it is more effective to take $\mathcal{G} = (\mathbb{Q}, +)$.

Remark 3.7. *$K((t))_{\mathrm{Puis};\mathcal{G}}$ has an interesting subring, the **semiring of bounded Puiseux series** $K((t))_{\mathrm{Puis}';\mathcal{G}}$ is*

$$\left\{ p(t) := \sum_{i \in \mathbb{Q},\ c_i \in K} c_i t^i \right\},$$

now restricting our sets I to range only over well-ordered subsets of $\frac{1}{n}\mathbb{Z}$ for $n \in \mathbb{N}$ depending on p; these have the form $\{\frac{m}{n} : m \geq m_0,\ n \text{ fixed}\}$. Note that $p(t^n)$ then is a usual Laurent series. (These are also the sets of fractional power series starting with a given m_0, in which the denominators are all bounded.)

Let us prove that $K((t))$ is indeed a field when K is a field. First we note that $K((t))$ is the localization $K[[t]][t^{-1}]$, so we consider $K[[t]]$.

Lemma 3.8. *An element $f = \sum_{i \in \mathbb{N}} c_i t_i \in K[[t]]$ is invertible iff c_0 is invertible.*

Proof. $(\Rightarrow)$ The constant term of $1 = ff^{-1}$ is a multiple of c.

$(\Leftarrow)$ Dividing through by c_0, we may assume that $c_0 = 1$, and write $f = 1 - tg$. It is easy to see that

$$f^{-1} = 1 + tg + t^2 g^2 + \dots$$

for $g \in K[[t]]$. The essence is to show that the right side defines a formal power series even though technically the sum is infinite and thus undefined. But for any m the coefficient of t^m in $t^j g^j$ is 0, for all $j > m$, each coefficient is just a finite sum in the obvious way. □

Remark 3.9. *Define $K[[t]]_{>n} = \{f = \sum_{i \in \mathbb{N}} c_i t_i \in K[[t]] : c_i = 0,\ \forall i \leq n\}$. (In particular $K[[t]]_{>0} = \{f \in K[[t]] : c_0 = 0\}$.) Then $K[[t]]_{>n} \triangleleft K[[t]]$.*

Proposition 3.10. *For any field K, let $K[[t]]_{>0} = \{f = \sum_{i \in \mathbb{N}} c_i t^i \in K[[t]] : c_0 = 0\}$ is invertible iff $c_0 \notin K[[t]]_{>0}$.*

Proof. This is a restatement of Lemma 3.8. □

Corollary 3.11. *$K((t))$ is a field.*

Proof. Any element $f \in K((t))$ can be written as $t^m f_0$, where $m \in \mathbb{Z}$ and f_0 has nonzero constant term and thus is invertible. But then $f^{-1} = t^{-m} f_0^{-1}$. □

Corollary 3.12. *$K((t))_{\mathrm{Puis}';\mathcal{G}}$ is a field.*

Proof. If $p \in K((t))_{\mathrm{Puis}';\mathcal{G}}$ then by definition, for some n, $p(t^n) \in K((t)))$, so $p(t^n)^{-1} \in K((t))$ and thus $p^{-1} \in K((t))_{\mathrm{Puis}';\mathcal{G}}$. □

The point of our discussion is:

Proposition 3.13. *If K is any algebraically closed field of characteristic 0 (customarily $\mathbb{C}$), then the field $K((t))_{\mathrm{Puis};\mathcal{G}}$ of Puiseux series is algebraically closed.*

This was proved over 100 years ago by means of extensive computations, but now can be seen as a consequence of valuation theory, so we defer the proof.

In the process of passing to Puiseux series, we have formalized the role of t, which could be thought as a sort of "generic" coefficient. From this point of view, passing to the tropical world is a process of "dequantizing," as elaborated by Litvinov in his overview, [L2].

3.4.1. *The order valuation.* Much of this material is extracted from [R1, Chapter 12].

Definition 3.14. *A **valuation** from an integral domain W to an ordered group $(\mathcal{G}, +)$ is a multiplicative monoid homomorphism $v : W \setminus \{0\} \to \mathcal{G}$, i.e., with*

$$v(ab) = v(a) + v(b),$$

*and satisfying the property $v(a + b) \geq \min\{v(a), v(b)\}$ for all $a, b \in K$. $\mathcal{G}$ is called the **value group** of v.*

*The valuation v is **trivial** if $v(W) = 0$.*

(More generally, $\mathcal{G}$ could be taken to be an ordered monoid, to be described below.) We formally put $v(0) = \infty$.

Remark 3.15. *(i) $v(1) = 0$ and, more generally, $v(a) = 0$ whenever $a^n = 1$ for some integer $n \neq 0$, since the only positive divisor $\frac{0}{n}$ is 0. In particular, $v(-1) = 0$.*

(ii) $v(-a = v(-1) + v(a) = v(a)$, by (i).

(iii) $v(n \cdot 1) \geq v(1) = 0$.

Lemma 3.16. *If $v(a) \neq v(b)$ then $v(a+b) = \min\{v(a), v(b)\}$.*

Proof. Suppose $v(a) > v(b)$. We know $v(a+b) \geq \min\{v(a), v(b)\} = v(b)$. But $b = (a+b) + -a$, so $v(b) \geq \min\{v(a+b), v(-a)\}$, implying $v(b) \geq v(a+b)$ (since we are given $v(a) > v(b)$). ☐

Remark 3.17. *Any valuation v from an integral domain W to an ordered group $(\mathcal{G}, +)$ extends naturally to a valuation $\hat{v}$ from the field of fractions F of W to $\mathcal{G}$, defined by $\hat{v}\left(\frac{a}{b}\right) = v(a) - v(b)$, since*

$$\hat{v}\left(\frac{a_1}{b_1}\frac{a_2}{b_2}\right) = v(a_1) + v(a_2) - v(b_1) - v(b_2) = v(a_1) - v(b_1) + v(a_2) - v(b_2) = \hat{v}\left(\frac{a_1}{b_1}\right) + \hat{v}\left(\frac{a_2}{b_2}\right)$$

and likewise

$$\hat{v}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right) = v(a_1 b_2 + a_2 b_1) - v(b_1) - v(b_2) = \max\{\hat{v}\left(\frac{a_1}{b_1}\right), \hat{v}\left(\frac{a_2}{b_2}\right)\}.$$

Thus, we can pass to the case that W is a field. For motivation, we might recall the two basic nontrivial examples from valuation theory:

Example 3.18. *(i) Suppose W is a PID (principal integral domain), and $p \in W$ is a prime element. The* p**-adic** *valuation v_p on W is defined via $v(u) = 0$ for every invertible $u \in W$ and $v_p(p^k a) = k$, where a is relatively prime to p. The value group of v_p clearly is $(\mathbb{Z}, +)$.*

We can extend this to the p**-adic** *valuation v_p on the field of fractions F, is defined via $v_p\left(p^k \frac{m}{n}\right) = k$, where a, b are relatively prime to p. The value group of v_p remains $(\mathbb{Z}, +)$. The main two examples here are for $W = \mathbb{Z}$, in which case $F = \mathbb{Q}$ (and these are the only valuations on $\mathbb{Q}$), and for $W = F[t]$ and $F = F(t)$, the field of rational functions (and note that the restriction of v to F is trivial).*

(ii) For $W = F_0[t]$ the algebra of polynomials in the indeterminate t over a field F_0, let us consider all valuations v which restrict trivially to F_0. Note that

$$\left|\sum_{i=0}^{n} \alpha_i t^i\right| \geq \min_{\alpha_i \neq 0} v(\alpha_i t^i) = \min_{\alpha_i \neq 0} v(\alpha_i) + v(t^i\}) = \min_{\alpha_i \neq 0} v(t^i);$$

equality holds if $v(t) \neq 0$ since then the $v(t^i)$ are all distinct.

CASE I. $v(t) < 0$. Then $v(\sum_{i=0}^n \alpha_i t^i) = v(t^n) = nv(t)$ (presuming $\alpha_n \neq 0$). Thus the value of a polynomial depends only on the degree of the polynomial.

CASE II $v(t) \geq 0$. Then every polynomial has value ≥ 0. The set of polynomials having value > 0 is a prime ideal of $F[t]$, and thus is generated by some irreducible polynomial p. If $p = 0$ then $v(\ \ )$ is trivial; if $p \neq 0$ then $v(\ \ )$ is a p-adic valuation.

Remark 3.19. *In Example 3.18, $v(t) > 0$ iff $p = t$, in which case $v(\sum_{i=u}^n \alpha_i t^i) = uv(t)$ (presuming $\alpha_u \neq 0$). Normalizing so that $v(t) = 1$, we get $v(t^u) = u$, the exponent of the lowest ordered term. This is the motivation for the* ***order valuation*** *which is so important in the sequel, because the completion of $F[t]$ under this valuation is $F[[t]]$.*

Here is the main example for the tropical theory reminiscent of Example 3.18.

Example 3.20. *For any integral domain K, the algebra $K((\lambda))_{\mathrm{Puis};\mathcal{G}}$ of Puiseux series has the* ***order valuation*** *v given by*

$$v(p(t)) \ := \min\{i \in G \ : \ c_i \neq 0\}.$$

We started out with the max-plus algebra, but Lemma 3.16 gives us $v(a+b) = \min\{v(a), v(b)\}$ when $v(a) \neq v(b)$ Accordingly, we replace v by $-v$ in order to switch back from minimum to maximum. But $-v(t) = v(1/t)$, and the dominant term in $p(1/t)$ becomes $c_{v(p(1/t))} t^{v(p(1/t))}$.

Remark 3.21. *The logarithm is just* $\log_t(c_{v(p(1/t))}) + v(p(1/t))$, *whose limit is* $v(p(1/t)$, *so in this sense, for* $K = \mathbb{C}$, *the logarithmic procedure described above boils down to taking the order valuation. There is one significant difference, however. When two Puisseux series have the same value, they have the form* $p = c_{i_0}t^{i_0} + \dots$ *and* $q = c'_{i_0}t^{i_0} + \dots$. *If* $c \neq -c'$, *then* $p+q = (c+c')t^{i_0} + \dots$, *and* $v(p+q) = i_0 = v(p) = v(q)$. *However, if* $c = -c'$ *then* $v(p+q) > i_0$, *and we have left the max-plus setting. (Compare with Remark 3.2.)*

Whereas logarithms often do not meld well with algebraic structure, valuation theory plays a crucial role in arithmetic and algebraic geometry; c.f. [Nag, M]. Thus, tropicalists have turned increasingly to the algebra of Puiseux series, and we need a way to reconcile the discrepancies of Remark 3.21 and Remark 3.2 from the max-plus algebra.

3.5. **Ultrafilters.** An alternative approach comes from logic. Let us dualize the notion of an ideal.

Definition 3.22. *A* ***filter*** *on a set* S *is a collection o*$\mathcal{F}$ *of subsets of* S *satisfying the following properties:*

- $S \in \mathcal{F}$ *but* $\emptyset \notin \mathcal{F}$;
- *If* $A, B \in \mathcal{F}$ *then* $A \cap B \in \mathcal{F}$;
- *If* $A \in \mathcal{F}$ *and* $B \supset A$ *then* $B \in \mathcal{F}$.

A maximal filter $\mathcal{F}$ *is called an* ***ultrafilter***.

Remark 3.23. *A filter cannot contain both* A *and its set complement* A', *for then it would contain* $A \cap A' = \emptyset$.

Example 3.24. *Suppose* S *is any infinite set. The collection of cofinite subsets of* S *(i.e., the complement is finite) is a filter, since the intersection of two cofinite sets is cofinite and thus nonempty.*

Remark 3.25. *The union of a chain of filters is a filter. Hence, by "Zorn's lemma," any filter is contained in an ultrafilter. (One cannot describe an ultrafilter containing the cofinite filter, but it is enough to know for us that it exists.)*

Here is a nice characterization of ultrafilters:

Lemma 3.26. *A filter* $\mathcal{F}$ *is an ultrafilter if and only if it satisfies the property:*
For any subset A *of* S, $\mathcal{F}$ *contains either* A *or* A' *(but not both).*

Proof. Clearly the condition is sufficient, since this makes $\mathcal{F}$ maximal possible. Conversely, say $\mathcal{F}$ is an ultrafilter and $A \notin \mathcal{F}$. We want to prove that $A' \in \mathcal{F}$. We take $\tilde{\mathcal{F}} = \mathcal{F} \cup \{(A' \cap B) : B \in \mathcal{F}\}$. $\tilde{\mathcal{F}}$ is a filter. (Otherwise, some $A' \cap B = \emptyset$, which would imply $A \supset B$ and thus $A \in \mathcal{F}$, contrary to hypothesis.) But this means $\tilde{\mathcal{F}} = \mathcal{F}$, implying $A' = A' \cap S \in \mathcal{F}$. □

Suppose now that we have a set of algebras $\{R_s : s \in S\}$ indexed by the elements of S. Given any filter $\mathcal{F}$ on S, we can define the **filtered product** $R_{\mathcal{F}} := (\prod_{s \in S} R_s)/\mathcal{F}$ to be the direct product $\prod_{s \in S} R_s$ modulo the relation $(a_s) \equiv (b_s)$ iff $\{s \in S : a_s = b_s\} \in \mathcal{F}$. It is easy to see that this is an equivalence, and furthermore an elementary sentence P (from the context of first-order logic) holds in $R_{\mathcal{F}}$ iff

$$\{s \in S : P \text{ holds in } R_s\} \in \mathcal{F}.$$

Example 3.27. *Take* $S = \{$*The prime integers*$\}$, *and construct the ultraproduct* $\tilde{R} = (\prod R_s)/\mathcal{F}$ *with respect to an arbitrary ultrafilter containing the cofinite filter on* S. *For each* $p \in S$, *let* $R_s = \mathbb{Z}/p\mathbb{Z}$ *Since each* R_s *is a field, so is* $\tilde{R}$. *But for each prime* p, *only one component has characteristic* p. *Thus, this field has characteristic 0, and does not contain any of the component fields.*

Example 3.28. *Take* $S = \mathbb{N}$, *and construct the ultraproduct* $\tilde{R} = (\prod R_s)/\mathcal{F}$ *with respect to an arbitrary ultrafilter containing the cofinite filter on* S. *For each* $s \in \mathbb{N}$, *let* $R_s = \mathbb{R}$. *The image in* $\tilde{R}$ *of* $(1, \frac{1}{2}, \frac{1}{3}, \dots)$ *is smaller than* $\frac{1}{n}$ *for each* n, *so* $\tilde{R}$ *contains "infinitesimally small" elements, as well as "infinitely large" elements such as the image of* $(1, 2, 3, \dots)$. *Let* $\hat{R}$ *be the subring of* $\tilde{R}$ *generated by the images of all* $\{(a_s) : \{|a_s : s \in \mathbb{N}\}$ *is bounded* $\}$. *The infinitesimal elements are an ideal* I *of* $\hat{R}$, *and thus* $\hat{R}/I$ *does not contain any infinitesimal elements.*

After this brief excursion into model theory, we can tie spines into the theory of ultraproducts.

Example 3.29. *Take* $S = \mathbb{N}$. *For each* $s \in \mathbb{N}$, *let* $R_s = (\mathbb{R}, \odot_s, \oplus_s)$, *and construct the ultraproduct* $\tilde{R} = (\prod R_s)/\mathcal{F}$ *with respect to an arbitrary ultrafilter containing the cofinite filter on* S. *Then defining* $\oplus$ *and* $\otimes$ *define operations that restrict componentwise to well-defined operations on* R *and it is easy to see that* *addition is the maximum, up to an infinitesimal (Indeed, $\otimes$ is clearly addition, since it is addition on each component, and one shows that for any ϵ that the difference $a \oplus_s b - \max\{a,b\} < \epsilon$ for almost all s. Thus, to obtain a max-plus algebra we should factor out by the infinitesimal elements, as in Example 3.28.*

Likewise, the Hausdorff distance from the spine to the "ultra-amoeba" is less than ϵ for each ϵ), so we have reconstructed the spine.

4. Elements of category theory and universal algebra

It is convenient to have an appropriate language in which we can discuss the different mathematical contexts in which we are working. From this point of view, we need some basic category theory. Category theory provides a powerful method to study the basic concepts of algebraic theories.

4.1. Preliminaries about categories.

Definition 4.1. *A* **category** $\mathcal{C}$ *is comprised of* **objects** *together with a set* $Hom_{\mathcal{C}}(A,B)$ *of morphisms (usually written* $Hom(A,B)$*) for any pair* A,B *of objects of* $\mathcal{C}$*, together with a* **composition**

$$Hom(B,C) \times Hom(A,B) \to Hom(A,C),$$

written $(g,f) \mapsto gf$ *(in which case we say the morphisms* g,f *are* **compatible***) satisfying the following two rules:*

(i) Associativity of composition: $(hg)f = h(gf)$ *(whenever the relevant morphisms are compatible);*

(ii) Each $Hom(A,A)$ *has a* **unit morphism** 1_A *satisfying* $g1_A = 1_B g = g$ *for all* $g \in Hom(A,B)$*.*

For example, **Set** denotes the category whose objects are the sets and whose morphisms $Hom(A,B)$ are the functions from A to B. But recall that the class of all sets is not a set! Whereas the categories that behave like **Set** attract our main attention, in each case the class of objects is not a set. Accordingly, we say a category is **small** if its objects comprise a set.

Example 4.2. *i) Any monoid* M *defines a small category, which has a single object* $*$*, and whose set of morphisms* $Hom(*,*) = M$*; composition is just the monoid multiplication in* M*. (The rules of composition are automatically satisfied, where the unit element of* M *takes the role of the unit morphism.)*

(ii) Any poset $(\mathcal{S}, \prec)$ *defines a small category, whose objects are the elements of* $\mathcal{S}$*, with* $Hom(a,b)$ *either a set with a single element or the empty set, depending on whether or not* $a \preceq b$*. Associativity of composition follows from transitivity of the relation.*

(iii) (Special case of (ii).) Any set I *can be endowed with the trivial PO (that all distinct elements are incomparable), and thus constitutes a small category, where* $Hom(a,b)$ *is* $\emptyset$ *whenever* $a \neq b$*, and* $Hom(a,a)$ *consists of the single morphism* 1_a*. Thus two distinct morphisms are never compatible, so the categorical axioms hold almost vacuously.*

(iv) Any topological space X *is a small category, whose objects are the open sets and whose morphisms are the inclusions. This can also be viewed as a special case of (ii), where* $\mathcal{S}$ *is the collection of open sets of* X*, ordered by inclusion. A common instance of this is when* X *has the discrete topology, so that every subset is open.*

A category $\mathcal{D}$ is called a **subcategory** of $\mathcal{C}$ if all objects of $\mathcal{D}$ are objects of $\mathcal{C}$ and $Hom_{\mathcal{D}}(A,B) \subseteq Hom_{\mathcal{C}}(A,B)$ for all objects A,B of $\mathcal{D}$. If $Hom_{\mathcal{D}}(A,B) = Hom_{\mathcal{C}}(A,B)$ for all objects A,B of $\mathcal{D}$, we say $\mathcal{D}$ is a **full subcategory** of $\mathcal{C}$.

Since the class of objects need not be a set, we are not permitted to use the laws of set theory with all the objects at once, and are drawn instead to $Hom_{\mathcal{C}}(A,B)$ and the morphisms. This is the great innovation of category theory, and from the outset we define concepts in terms of "arrows," often as they appear in commutative diagrams. We write $f: A \to B$ to designate the morphism f in $Hom(A,B)$.

An **isomorphism** is just a morphism $f: A \to B$ for which there is a morphism $g: B \to A$ such that $gf = 1_A$ and $fg = 1_B$. However, it is less obvious how to categorize the notions of 1:1 and onto.

Definition 4.3. *(i) A morphism* $f: A \to B$ *is* **monic** *if for any morphisms* $g \neq h: C \to A$ *we have* $fg \neq fh$*.*

(ii) A morphism $f: A \to B$ *is* **epic** *if for any* $g \neq h: B \to C$ *we have* $gf \neq hf$*.*

To some extent, monics correspond to 1:1 maps, and epics to correspond to onto maps.

Remark 4.4. *If $\mathcal{C}$ is a subcategory of* **Set**, *then any 1:1 map is monic, and any onto map is epic. (Indeed, if $f : A \to B$ is 1:1 and $g \neq h : C \to A$, then taking $c \in C$ such that $g(c) \neq h(c)$, we have $fg(c) \neq fh(c)$, so $fg \neq fh$. Likewise if f is onto and $g \neq h : B \to C$ then taking $g(b) \neq h(b)$ and writing $b = f(a)$ we see $gf(a) \neq hf(a)$.)*

The converse also holds in **Set**, seen by reversing the above arguments, but does not always hold in general.

Definition 4.5. *The* **dual category** $\mathcal{C}^{\mathrm{op}}$ *of a category $\mathcal{C}$ has the same objects as C, but the morphisms are obtained by reversing the arrows, i.e., $Hom(A, B)$ in $\mathcal{C}^{\mathrm{op}}$ is $Hom(B, A)$ in $\mathcal{C}$; likewise the composition is reversed. (If $f \in Hom_{\mathcal{C}^{\mathrm{op}}}(B, C)$ and $g \in Hom_{\mathcal{C}^{\mathrm{op}}}(A, B)$ then f, g correspond respectively to morphisms in $Hom_{\mathcal{C}}(C, B)$ and $Hom_{\mathcal{C}}(B, A)$ that may be compatible in reverse order.)*

Example 4.6. *(i) Viewing a monoid M as a small category, we define M^{op} to be the same set, but with multiplication in reverse order.*

(iii) Viewing a poset $(\mathcal{S}, \prec)$ as a small category, we see its dual category is the dual poset. In particular, the dual lattice is the categorical dual of a lattice (viewed as a small category).

The duals of non-small categories are harder to describe. On the other hand, the dual of an interesting categorical notion tends to be interesting. For example, the dual of "monic" is "epic." Often we define a dual notion formally, appending the prefix "co;" the following definition is in this spirit.

There are many other situations in which two categories have the same theory, but we do not utilize them at present. The prototypical category from algebra is the category of modules over a ring, and here $Hom(A, B)$ is an Abelian group since we can negate the module homomorphisms. This does not work in the tropical situation, and will require considerable care.

4.2. **Universal algebra.** We present our categories in the framework of universal algebra. We write $A^{(n)}$ for the direct product $A \times \cdots \times A$ of a set A, taken n times. An **algebraic structure** is a set A, endowed with a **signature**, which is set $\Omega = \{\omega_{n,i} : n \in \mathbb{N}, 0 \leq i \leq m_n\}$ of various operations $\omega_{n,i} : A^{(n)} \to A$, called **$n$-ary products**, as well as **universal relations**, which can be written as formulas $f(\mathbf{x}) = g(\mathbf{x})$, where f and g are expressions in the operations and $\mathbf{x}$ denotes a set of indeterminates; by this we mean the universal sentence $f(\mathbf{a}) = g(\mathbf{a})$ for all $a_i \in A$. One defines homomorphisms as maps preserving the products in the obvious way, and we thereby get a category of Ω-algebras. A **substructure** is defined in the usual way as a 1:1 homomorphism.

We work within the given signature Ω, which defines our category, and call A a **carrier** for Ω. Often the signature Ω is finite.

Certain constructions fit in very well with the theory. The direct product of carriers is a carrier, and combining these with homomorphic images, we see that the ultraproduct of carriers is a carrier. Thus, model theory often works in harmony with universal algebra.

Example 4.7. *The categories arising in universal algebra are the subcategories of* **Set** *whose objects are structures of a given signature, and the morphisms are the homomorphisms in that signature. These categories are not small. For example, we have*

- **Semigroup** *is the category whose objects are semigroups, and $Hom(A, B)$ is the set of semigroup homomorphisms $f : A \to B$; i.e., $f(a_1 a_2) = f(a_1)f(a_2)$, $\forall a_1, a_2 \in A_1$. We need a 2-ary product $\omega_{2,0}$ for the monoid operation, together with the relation given by the formula $(x_1x_2)x_3 = x_1(x_2x_3)$.*
- **Mon** *is the category whose objects are monoids, and $Hom(A, B)$ is the set of semigroup homomorphisms $f : A \to B$ satisfying $f(1) = 1$. We need an 0-ary product $\omega_{0,1}$ for the distinguished element 1, as well as a 2-ary product $\omega_{2,0}$ for the monoid operation, together with the relations given by the formulas $x \cdot 1 = x$, $1 \cdot x = x$, and $(x_1x_2)x_3 = x_1(x_2x_3)$.*
- **Grp** *is the category whose objects are groups, and whose morphisms are group homomorphisms. In addition to the previous products, we need a 1-ary product $\omega_{1,0}$ for inversion $a \mapsto a^{-1}$.*

  Grp *is a subcategory of* **Mon**.
- **Ab** *is the full subcategory of* **Grp** *whose objects are Abelian groups. We need the relations given by the formula $x_1 + x_2 = x_2 + x_1$.*

  Semiring† *is the subcategory of* **Mon** *whose objects are semirings†, and whose morphisms are homomorphisms both additively and multiplicatively. We need a 0-ary product $\omega_{0,0}$ for the distinguished elements 1, and 2-ary products $\omega_{2,0}$ and $\omega_{2,1}$ for addition and multiplication, together with* *the relations given by the formulas* $x \cdot 1 = x$, $1 \cdot x = x$, $x_1 + x_2 = x_2 + x_1$, $(x_1+x_2)+x_3 = x_1+(x_2+x_3)$, *and distributivity.*

- **Ring** *is the full subcategory of* **Ab** *whose objects are rings (say with unit element 1), and whose morphisms are ring homomorphisms. We need 0-ary products* $\omega_{0,0}$ *and* $\omega_{0,1}$*) for the distinguished elements 0 and 1, a 1-ary product* $\omega_{1,0}$ *for negation* $a \mapsto -a$*, and 2-ary products* $\omega_{2,0}$ *and* $\omega_{2,1}$ *for addition and multiplication, together with the relations given by the formulas* $x+(-x) = 0$, $x+0 = x$, $x \cdot 1 = x$, $1 \cdot x = x$, $x_1 + x_2 = x_2 + x_1$, $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3)$, *and distributivity.*
- *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are categories then one can define the category* $\mathcal{C} \times \mathcal{D}$*, whose objects are pairs* (C, D) *where* C *and* D *are objects of* $\mathcal{C}$ *and* $\mathcal{D}$ *respectively, and with*

$$Hom_{\mathcal{C}\times\mathcal{D}}((C_1, D_1),(C_2, D_2)) = Hom_{\mathcal{C}}(C_1, C_2) \times Hom_{\mathcal{D}}(D_1, D_2).$$

4.2.1. *Congruences.*

A **congruence** Φ of an algebra A is an equivalence relation $\equiv$ that preserves all the relevant operations and relations; we call $\equiv$ the **underlying equivalence** of Φ. Thus, the congruence Φ may be viewed either as a subalgebra of $A \times A$ containing the diagonal and satisfying two other conditions corresponding to symmetry and transitivity (in which case we describe Φ as the appropriate set of ordered pairs, or otherwise Φ may be viewed as an equivalence relation $\equiv$ satisfying certain algebraic conditions. We utilize both approaches, relying on the context to make the notation clear.

Remark 4.8. *We recall some key results of* [J, §2]*:*

(i) *Given a congruence* Φ *of an algebraic structure* A*, one can endow the set*

$$A/\Phi := \{[a] : a \in A\}$$

of equivalence classes with the same (well-defined) algebraic structure, and the map $a \mapsto [a]$ *defines an onto homomorphism* $A \to A/\Phi$*.*

(ii) *In the opposite direction, for any homomorphism* $\varphi : A \to A'$*, one can define a congruence* Φ_φ *on* A *by saying that*

$$a \equiv_\varphi b \quad \text{iff} \quad \varphi(a) = \varphi(b).$$

We call Φ_φ *the* ***congruence of*** φ*. Then* φ *induces a 1:1 homomorphism* $\widetilde{\varphi} : A/\Phi_\varphi \to A'$*, via* $\widetilde{\varphi}([a]) = \varphi(a)$*, for which* φ *factors through*

$$A \to A/\Phi_\varphi \to A',$$

as indicated in [J, p. 62]*. Thus the homomorphic images of* A *correspond to the congruences defined on* A*.*

In the case of ring theory (or likewise module theory), the congruence is determined by the kernel of the homomorphism, since $[a] = [a']$ iff $[a - a'] = [0]$. But lacking the negation operation, we cannot make this reduction for semirings†, and thus the usual factor construction with respect to ideals does not work for us in general. In view of Remark 4.8, congruences take the place of ideals (from ring theory) in defining homomorphisms.

4.3. Identification of categories: Ordered Abelian monoids versus commutative semirings†.

4.3.1. *Functors.* Having shifted emphasis to categories, we would like a way to compare two categories.

Definition 4.9. *A* **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ *from a category* $\mathcal{C}$ *to a category* $\mathcal{D}$ *is a correspondence sending each object* A *of* $\mathcal{C}$ *to an object* FA *of* $\mathcal{D}$*, and each morphism* $f : A \to B$ *of* $\mathcal{C}$ *to a morphism* $F(f) : FA \to FB$ *in* $\mathcal{D}$*, such that* $F(1_A) = 1_{F(A)}$ *for every object* A *of* $\mathcal{C}$ *and*

$$F(fg) = F(f)F(g)$$

whenever f, g *are compatible morphisms.*

A **contravariant functor** *from* $\mathcal{C}$ *to* $\mathcal{D}$ *is a covariant functor from* $\mathcal{C}^{\mathrm{op}}$ *to* $\mathcal{D}$*.*

Example 4.10. *(i) The* **identity functor** $F = 1_{\mathcal{C}}$ *satisfies* $FA = A$ *and* $F(f) = f$*, for every object* A *and every morphism* f*.*

(ii) The **forgetful functor** *"forgets" some of the structure in universal algebra. For example* $F : \mathbf{Grp} \to \mathbf{Mon}$ *views a group as a monoid, forgetting that elements have inverses.*

Definition 4.11. *Two categories $\mathcal{C}$, $\mathcal{D}$ are* **isomorphic** *if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$.*

There is a well-known localization procedure with respect to a submonoid S of a cancellative Abelian monoid $\mathcal{M}$, obtained by taking $\mathcal{M} \times S/\sim$, where $\sim$ is the equivalence relation given by $(a, s) \sim (a', s')$ iff $as' = a's$. Localizing with respect to all of $\mathcal{M}$ yields its **group of fractions**, cf. [Bo, W]. We say that a monoid $\mathcal{M}$ is **power-cancellative** (called **torsion-free** by [W]) if $a^n = b^n$ for some $n \in \mathbb{N}$ implies $a = b$. A monoid $\mathcal{M}$ is called $\mathbb{N}$**-divisible** (also called **radicalizible** in the tropical literature) if for each $a \in \mathcal{M}$ and $0 < m \in \mathbb{N}$ there is $b \in \mathcal{M}$ such that $b^m = a$. For example, $(\mathbb{Q}, +)$ is $\mathbb{N}$-divisible.

Remark 4.12. *The customary way of embedding an Abelian monoid $\mathcal{M}$ into an $\mathbb{N}$-divisible monoid, is to adjoin $\sqrt[m]{a}$ for each $a \in \mathcal{M}$ and $m \in \mathbb{N}$, and define*

$$\sqrt[m]{a}\,\sqrt[n]{b} := \sqrt[mn]{a^n b^m}.$$

This will be power-cancellative if $\mathcal{M}$ is power-cancellative.

The passage to the max-plus algebra in tropical mathematics can be viewed algebraically via ordered groups (such as $(\mathbb{R}, +)$), and, more generally, ordered monoids.

Remark 4.13. *Any ordered Abelian monoid becomes a commutative bipotent semiring† when we define $a + b = \max\{a, b\}$. Conversely, any proper semiring† R gives rise to a partial order, given by $a \le b$ iff $a + c = b$ for some $c \in R$. This is a total order when the semiring† R is bipotent.*

Stated more precisely, the categories of ordered Abelian monoids and bipotent commutative semirings† are isomorphic.

We can of course go back and forth.

Remark 4.14. *Given a commutative bipotent domain† R, we take the group of fractions of its multiplicative monoid, which, being ordered, can be viewed as a semiring† and by definition is a semifield†. Thus, every commutative bipotent domain† had a semifield† of fractions.*

Each language has its particular advantages. One advantage of working with ordered monoids and groups is that their elementary theory is well-known to model theorists. The theory of ordered $\mathbb{N}$-divisible ordered Abelian groups is model complete, cf. [M, p. 116] and [Sa, pp. 35,36], which essentially means that every $\mathbb{N}$-divisible ordered cancellative Abelian monoid has the same algebraic theory as the max-plus algebra $(\mathbb{Q}, +)$, which is a much simpler structure than $(\mathbb{R}, +)$. From this point of view, the algebraic essence of tropical mathematics boils down to $(\mathbb{Q}, +)$, and for this reason we prefer to work with Puiseux series with exponents in $\mathbb{Q}$. Sometimes we want to study its ordered submonoid $(\mathbb{Z}, +)$, or even $(\mathbb{N}, +)$, although they are not $\mathbb{N}$-divisible.

On the other hand, the two operations of semirings† enable us to construct matrices and polynomials semirings†, which are so important in linear algebra and geometry and become the focus of our study.

4.4. **Kernels of semifields†.** Although congruences in semirings† provide a theoretical replacement for ideals in the structure theory, they are much more difficult to investigate. Fortunately, there is a significant simplification for semifields†, and thus for domains† by means of Remark 4.14.

Definition 4.15. *A* ***kernel*** *of a semifield† F is a subgroup K which is convex in the sense that if $a, b \in K$ and $\alpha, \beta \in F$ with $\alpha + \beta = \mathbb{1}_F$, then $\alpha a + \beta b \in K$.*

We have the following key correspondence. Given a congruence Φ on a domain† R with semifield† of fractions F, we define $K_\Phi = \{\frac{a}{b} \in F : a, b \in R \text{ and } a \equiv b\}$. (Or, in other words, extending Φ to F as in Lemma **??**, we require $\frac{a}{b} \equiv \mathbb{1}_F$.)

Theorem 4.16. *There is a 1:1 correspondence between prime congruences of a domain† R and kernels of F, given by $\Phi \to K_\Phi$. Any homomorphism of domains† $R \to R'$ gives rise to a kernel, and $Cong$ is the congruence corresponding to $R \mapsto R/Cong$.*

Proof. We extend Φ to F as in Lemma **??**. As noted in [**?**, Theorem 3.2], K_Φ is clearly a congruence, since $\alpha a + \beta bb \equiv \alpha + \beta \equiv \mathbb{1}_F$.

In the other direction, given a kernel K of F, we define the congruence Φ on F according to [**?**, Definition 3.1], i.e., $a \equiv b$ iff $\frac{a}{b} \equiv \mathbb{1}_F$, and restrict this to R.

Given a homomorphism $R \to R'$, we compose it with the injection of R' into its semifield† of fractions F', and then extend this naturally to a homomorphism of semifields† $F \to F'$, thereby obtaining a kernel. $\square$

5. Polynomials and their roots

Since algebraic geometry begins with polynomials, a tropical geometric theory requires a careful treatment of polynomials over semirings†.

5.1. The function semiring†.

Definition 5.1. *The* ***function semiring***† $\mathrm{Fun}(S,R)$ *is the set of functions from a set* S *to a semiring*† R.

$\mathrm{Fun}(S,R)$ becomes a semiring† under componentwise operations, and is proper when R is proper. Customarily one takes $S = R^{(n)}$, the Cartesian product of n copies of R. This definition enables us to work with proper subsets, but the geometric applications lie outside the scope of the present paper.

5.2. Polynomials as functions.

$\Lambda = \{\lambda_1, \dots, \lambda_n\}$ always denotes a finite set of indeterminates commuting with the semiring† R; often $n = 1$ and we have a single indeterminate λ. We have the polynomial semiring† $R[\Lambda]$. As in [IzR1], we view polynomials in $R[\Lambda]$ as functions, but perhaps viewed over some extension R' of R. More precisely, for any subset $S \subseteq R^{(n)}$, there is a natural semiring† homomorphism

$$\psi : R[\Lambda] \to \mathrm{Fun}(S,R), \tag{5.1}$$

obtained by viewing a polynomial as a function on S.

When R is a semifield†, the same analysis is applicable to Laurent polynomials $R[\Lambda,\Lambda^{-1}]$, since the homomorphism $\lambda_i \mapsto a_i$ then sends $\lambda_i^{-1} \mapsto a_i^{-1}$. Likewise, when R is power-cancellative and divisible, we can also define the **semiring† of rational polynomials** $R[\Lambda]_{\mathrm{rat}}$, where the powers of the λ_i are taken to be arbitrary rational numbers. These can all be viewed as elementary formulas in the appropriate languages, so the model theory alluded to earlier is applicable to the appropriate polynomials and their (tropical) roots in each case.

Other functions over the bipotent semiring† R of an ordered monoid $\mathcal{M}$ can be defined in the same way. For example, we say that an ordered monoid $\mathcal{M}$ is **archimedean** if for each $a \in \mathcal{M}$ there is some natural number $m \in \mathbb{N}$ such that $a < m$. Then we can define the formal exponential series

$$\exp(a) := \sum_k \frac{a^k}{k!} \tag{5.2}$$

since $a < m$ implies $\frac{a^{m+1}}{(m+1)!} < \frac{a^m}{m!}$, and thus (5.2) becomes a finite sum. It follows at once that $\exp(\lambda) := \sum \frac{\lambda^k}{k!}$ is defined in $\mathrm{Fun}(R,R)$.

5.3. Tropicalization.

5.4. Tropical roots.

5.4.1. *Statement of Kapranov's Theorem.*

6. Valuations

6.1. Basic theory.

Recall the definition of valuation 3.14.

The following basic observation in valuation theory shows why valuations are relevant to the tropical theory.

Remark 6.1. *If* $v(a) \neq v(b)$, *then* $v(a+b) = \min\{v(a), v(b)\}$. *Inductively, if* $v(a_1), \dots, v(a_m)$ *are distinct, then*

$$v\left(\sum_{i=1}^m a_i\right) = \min\{v(a_i) : 1 \le i \le m\} \in \mathcal{M}.$$

Consequently, if $\sum a_i = 0$, *then at least two of the* $v(a_i)$ *are the same. These considerations are taken much more deeply in* [BiG].

When W is a field, the value monoid $\mathcal{G}$ is a group. Much information about a valuation $v : W \to \mathcal{G} \cup \{\infty\}$ can be garnered from the target $v(W)$, but valuation theory provides some extra structure:

- The **valuation ring** $O_v = \{a \in W : v(a) \ge 0\}$,
- The **valuation ideal** $P_v = \{a \in W : v(a) > 0\}$,

- The **residue ring** $\bar{W} = O_v/P_v$, a field if W is a field.

For example, the valuation ring of the order valuation on the field $\mathbb{K}$ of Puiseux series is $\{p(t) \in \mathbb{K} : c_i = 0$ for $i < 0\}$, and the residue field is K.

We replace v by $-v$ to switch minimum to maximum, and ∞ by $-\infty$. One can generalize the notion of valuation to permit W to be a semiring†; taking $W = \mathcal{M}$, we see that the identity map is a valuation, which provides one of our main examples.

6.2. **Puiseux Series revisited.**

6.3. **Proof of Kapranov's Theorem.**

6.4. **Ordered groups and monoids.**

6.5. **The standard supertropical semiring†.** This construction, following [IzR1], refines the max-plus algebra and picks up the essence of the value monoid. From now on, in the spirit of max-plus, we write the operation of an ordered monoid $\mathcal{M}$ as multiplication.

We start with an Abelian monoid $\mathcal{M} := (\mathcal{M}, \cdot)$, an ordered group $\mathcal{G} := (\mathcal{G}, \cdot)$, and an onto monoid homomorphism $v : \mathcal{M} \to \mathcal{G}$. We write a^ν for $v(a)$, for $a \in \mathcal{M}$. Thus every element of $\mathcal{G}$ is some a^ν. We write $a \cong_\nu b$ if $a^\nu = b^\nu$.

Our two main examples:

- $\mathcal{M} = \mathcal{G}$ is the ordered monoid of the max-plus algebra (the original example in Izhakian's dissertation);
- $\mathcal{M}$ is the multiplicative group of a field F, and $v : F^\times \to \mathcal{G}$ is a valuation. Note that we forget the original addition on the field F!

Our object is to use the ordering on $\mathcal{G}$ to study $\mathcal{M}$. Accordingly we want to define a structure on $\mathcal{M} \cup \mathcal{G}$.

The **standard supertropical semiring†** R is the disjoint union $\mathcal{M} \cup \mathcal{G}$, made into a monoid by starting with the given multiplications on $\mathcal{M}$ and $\mathcal{G}$, and defining $a \cdot b^\nu$ and $a^\nu \cdot b$ to be $(ab)^\nu$ for $a, b \in \mathcal{M}$. We extend v to the **ghost map** $\nu : R \to \mathcal{G}$ by taking $\nu|_\mathcal{M} = v$ and $\nu|_\mathcal{G}$ to be the identity on $\mathcal{G}$. Thus, ν is a monoid projection.

We make R into a semiring† by defining

$$a + b = \begin{cases} a & \text{for } a^\nu > b^\nu; \\ b & \text{for } a^\nu < b^\nu; \\ a^\nu & \text{for } a^\nu = b^\nu. \end{cases}$$

R is never additively cancellative (except for $\mathcal{M} = \{\mathbb{1}\}$).

$\mathcal{M}$ is called the **tangible submonoid** of R. $\mathcal{G}$ is called the **ghost ideal**.

R is called a **supertropical domain†** when the monoid $\mathcal{M}$ is (multiplicatively) cancellative.

Strictly speaking, a supertropical domain† will not be a semifield† since the ghost elements are not invertible. Accordingly, we define a **1-semifield†** to be a supertropical domain† for which $\mathcal{M}$ is a group.

Motivation: The ghost ideal $\mathcal{G}$ is to be treated much the same way that one treats the zero element in commutative algebra. Towards this end, we write

$$a \underset{\text{gs}}{\models} b \qquad \text{if} \qquad a = b \quad \text{or} \quad a = b + \text{ghost}.$$

(Accordingly, write $a \underset{\text{gs}}{\models} \mathbb{0}$ if a is a ghost.) Note that for a tangible, $a \underset{\text{gs}}{\models} b$ iff $a = b$. If needed, we could formally adjoin a zero element in a separate component; then the ghost ideal is $\mathcal{G}_\mathbb{0} := \mathcal{G} \cup \{\mathbb{0}\}$. We may think of the ghost elements as uncertainties in classical algebra arising from adding two Puiseux series whose lowest order terms have the same degree.

R is a cover of the max-plus algebra of $\mathcal{G}$, in which we "resolve" tangible idempotence, in the sense that $a + a = a^\nu$ instead of $a + a = a$.

This modification in the structure permits us to detect corner roots of tropical polynomials in terms of the algebraic structure, by means of ghosts. Namely, we say that $\mathbf{a} \in R^{(n)}$ is a root of a polynomial $f \in R[\Lambda]$ when $f(\mathbf{a}) \in \mathcal{G}$. This concise formulation enables us to apply directly many standard mathematical concepts from algebra, algebraic geometry, category theory, and model theory, as described in [IzKR1]–[**?**] and [IzR1]–[IzR6].

The standard supertropical semiring works well with linear algebra, as we shall see.

6.6. **Kapranov's Theorem and the exploded supertropical structure.** Given a polynomial $f(\Lambda) = \sum_{\mathbf{i}} p_{\mathbf{i}}(\lambda_1^{i_1} \cdots \lambda_n^{i_n}) \in \mathbb{K}[\Lambda]$, i.e., with each $p_{\mathbf{i}}$ a Puiseux series, we define its **tropicalization** $\tilde{f}$ to be the tropical polynomial $\sum_{\mathbf{i}} v(p_{\mathbf{i}})\lambda_1^{i_1} \cdots \lambda_n^{i_n}$. (In the tropical literature, this is customarily written in the circle notation.) By Remark 6.1, if $\mathbf{a}$ is a root of f in the classical sense, then $v(\mathbf{a})$ is a tropical root of $\tilde{f}$. Kapranov showed, conversely, that any tropical root of $\tilde{f}$ has the form $v(\mathbf{a})$ for suitable $\mathbf{a} \in \mathbb{K}^{(n)}$, and valuation theory can be applied to give a rather quick proof of this fact, although we are not aware of an explicit reference. (See [R1, Proposition 12.58] for an analogous proof of a related valuation-theoretic result.) But the proof requires us to know more than just the lowest power of a Puiseux series, but also its coefficient; i.e., we also must take into account the residue field of the order valuation. Thus, we need to enrich the supertropical structure to include this extra information. This idea was first utilized by Parker [Par] in his "exploded" tropical mathematics. Likewise, Kapranov's Theorem has been extended by Payne [Pay1, Pay2], for which we need the following more refined supertropical structure, initiated by Sheiner [ShSh]:

Definition 6.2. *Given a valuation $v : W \to \mathcal{G}$, we define the* ***exploded*** *supertropical algebra $R = W \times \mathcal{G}$, viewed naturally as a monoid. (Thus we are mixing the "usual" world with the tropical world.)*

We make R into a semiring† by defining

$$(c, a) + (d, b) = \begin{cases} (c, a) & \text{when } a > b; \\ (d, b) & \text{when } a < b; \\ (c + d, a) & \text{when } a = b. \end{cases}$$

Sheiner's theory parallels the standard supertropical theory, where now the ghost elements are taken to be the 0-layer $\{0\} \times \mathcal{G}$.

7. The layered structure

The standard supertropical theory has several drawbacks. First, it fails to detect the multiplicity of a root of a polynomial. For example we would want 3 to have multiplicity 5 as a tropical root of the tropical polynomial $(\lambda + 3)^5$; this is not indicated supertropically. Furthermore, serious difficulties are encountered when attempting to establish a useful intrinsic differential calculus on the supertropical structure. Also, some basic supertropical verifications require ad hoc arguments.

These drawbacks are resolved by refining the ghost ideal into different "layers," following a construction of [WW, Example 3.4] and [AkGG, Proposition 5.1]. Rather than a single ghost layer, we take an indexing set L which itself is a partially ordered semiring†; often $L = \mathbb{N}$ under classical addition and multiplication.

Ordered semirings† can be trickier than ordered groups, since, for example, $a > b$ in $(\mathbb{R}, \cdot\,)$ does not imply $-a > -b$, but rather $-a < -b$. To circumvent this issue, we require all elements in the indexing semiring† to be non-negative.

Construction 7.1 ([IzKR4, Construction 3.2]). *Suppose we are given a cancellative ordered monoid $\mathcal{G}$, viewed as a semiring† as above. For any partially ordered semiring† L we define the semiring† $\mathscr{R}(L, \mathcal{G})$ to be set-theoretically $L \times \mathcal{G}$, where we denote the element (ℓ, a) as ${}^{[\ell]}a$; we define multiplication componentwise, i.e., for $k, \ell \in L$, $a, b \in \mathcal{G}$,*

$${}^{[k]}a \ {}^{[\ell]}b = {}^{[k\ell]}(ab), \tag{7.1}$$

and addition via the rules:

$${}^{[k]}a + {}^{[\ell]}b = \begin{cases} {}^{[k]}a & \text{if } a > b, \\ {}^{[\ell]}b & \text{if } a < b, \\ {}^{[k+\ell]}a & \text{if } a = b. \end{cases} \tag{7.2}$$

$R := \mathscr{R}(L, \mathcal{G})$ is indeed a semiring†. We identify $a \in \mathcal{G}$ with ${}^{[1]}a \in R_1$.

In most applications the "sorting" semiring† L is ordered, and its smallest nonzero element is 1. In this case, the monoid $\{ {}^{[\ell]}a : 0 < \ell \leq 1\}$ is called the **tangible** part of R. The **ghosts** are $\{ {}^{[\ell]}a : \ell > 1\}$, and correspond to the ghosts in the standard supertropical theory. The ghosts together with R_0 comprise an ideal. If there is a zero element it would be ${}^{[0]}\mathbb{0}$.

One can view the various choices of the sorting semiring† L as different stages of degeneration of algebraic geometry, where the crudest (for $L = \{1\}$) is obtained by passing directly to the familiar max-plus algebra. The supertropical structure is obtained when $L = \{1, \infty\}$, where R_1 and R_∞ are two copies of $\mathcal{G}$, with R_1

the tangible submonoid of $\mathcal{G}$ and R_∞ being the ghost copy. Other useful choices of L include $\{1,2,\infty\}$ (to distinguish between simple roots and multiple roots) and $\mathbb{N}$, which enables us to work with the multiplicity of roots and with derivatives, as seen below. In order to deal with tropical integration as anti-differentiation, one should consider the sorting semirings† $\mathbb{Q}_{>0}$ and $\mathbb{R}_{>0}$, but this is outside our present scope.

By convention, ${}^{[\ell]}\lambda$ denotes ${}^{[\ell]}\mathbb{1}_R\,\lambda$. Thus, any monomial can be written in the form $xl\alpha_{\mathbf{i}}\ell\lambda_1^{i_1}\cdots\lambda_n^{i_n}$ where $\mathbf{i}=(i_1,\dots,i_n)$. We say a polynomial f is **tangible** if each of its coefficients is tangible.

Note that the customary decomposition $R=\bigoplus_{\ell\in L}R_\ell$ in graded algebras has been strengthened to the partition $R=\dot{\bigcup}_{\ell\in L}R_\ell$. The ghost layers now indicate the number of monomials defining a corner root of a tangible polynomial. Thus, we can measure multiplicity of roots by means of layers. For example,

$$(\lambda+3)^5 = {}^{[1]}\lambda^5 + {}^{[5]}3\,\lambda^4 + {}^{[10]}6\,\lambda^3 + {}^{[10]}9\,\lambda^2 + {}^{[5]}12\,\lambda + {}^{[1]}15\,,$$

and substituting 3 for λ gives ${}^{[32]}15 = {}^{[2^5]}3^5$.

7.1. **Layered derivatives.** Formal derivatives are not very enlightening over the max-plus algebra. For example, if we take the polynomial $f=\lambda^2+5\lambda+8$, which has corner roots 3 and 5, we have $f'=2\lambda+5$, having corner root 3, but the common corner root 3 of f and f' could hardly be considered a multiple root of f. This difficulty arises from the fact that $1+1\neq 2$ in the max-plus algebra. The layering permits us to define a more useful version of the derivative (where now R contains a zero element $\mathbb{0}_R$):

Definition 7.2. *The **layered derivative** f'_{lay} of f on $R[\lambda]$ is given by:*

$$\left(\sum_{j=0}^{n} {}^{[\ell_j]}\alpha_j\,\lambda^j\right)'_{\mathrm{lay}} := \sum_{j=1}^{n} {}^{[j\ell_j]}\alpha_j\,\lambda^{j-1}. \tag{7.3}$$

In particular, for $\alpha = {}^{[1]}\alpha \in R_1$,

$$(\alpha\lambda^j)'_{\mathrm{lay}} := {}^{[j]}\alpha\,\lambda^{j-1} \quad (j\geq 2), \qquad (\alpha\lambda)'_{\mathrm{lay}} := \alpha, \qquad \text{and} \quad \alpha'_{\mathrm{lay}} := \mathbb{0}_R.$$

Thus, we have the familiar formulas:

(1) $(f+g)'_{\mathrm{lay}} = f'_{\mathrm{lay}} + g'_{\mathrm{lay}}$;

(2) $(fg)'_{\mathrm{lay}} = f'_{\mathrm{lay}}g + fg'_{\mathrm{lay}}$.

This is far more informative in the layered setting (say for $L=\mathbb{N}$) than in the standard supertropical setting, in which $(\alpha\lambda^j)'$ is ghost for all $j\geq 2$.

7.2. **The tropical Laplace transform.** The classical technique of Laplace transforms has a tropical analog which enables us to compare the various notions of derivative. Suppose L is infinite, say $L=\mathbb{N}$. Formally permitting infinite vectors $(a_\ell)_{\ell\in L}$ permits us to define a homomorphism $R[[\Lambda]]\to\mathscr{R}(L,R)$ given by

$$\sum a_k\lambda^k \mapsto \left({}^{[k]}k!\,a_k\right).$$

(Strictly speaking, we would want the image to be $({}^{[\frac{1}{k}]}k!\,a_k)$, but this would complicate the notation and require us to take $L=\mathbb{Q}^+$.) For example, $\exp_{\mathrm{lay}}(a)\mapsto(\,{}^{[k]}a_k\,)$ where each $a_k=a$.

Now we define $(\,{}^{[\ell]}a_\ell\,)' = (\,{}^{[\ell-1]}a_\ell\,)$. Then $\exp'_{\mathrm{lay}} = \exp_{\mathrm{lay}}$. This enables one to handle trigonometric functions in the layered theory.

7.3. **Layered domains† with symmetry, and patchworking.** Akian, Gaubert, and Guterman [AkGG, Definition 4.1] introduced an involutory operation on semirings, which they call a **symmetry**, to unify the supertropical theory with classical ring theory. One can put their symmetry in the context of $\mathscr{R}(L,\mathcal{G})$.

Definition 7.3. *A **negation map** on a semiring† L is a function $\tau: L\to L$ satisfying the properties:*

N1. $\tau(k\ell)=\tau(k)\ell=k\tau(\ell)$;
N2. $\tau^2(k)=k$;
N3. $\tau(k+\ell)=\tau(k)+\tau(\ell)$.

Suppose the semiring† L has a negation map τ of order ≤ 2. We say that $R := \mathscr{R}(L, \mathcal{G})$ has a **symmetry** σ when R is endowed with a map

$$\sigma : R \to R$$

and a negation map τ on L, together with the extra axiom:

S1. $s(\sigma(a)) = \tau(s(a)), \quad \forall a \in R.$

Example 7.4. *Suppose L is an ordered semiring†. We mimic the well-known construction of $\mathbb{Z}$ from $\mathbb{N}$. Define the* ***doubled semiring†***

$$D(L) = L_1 \times L_{-1},$$

the direct product of two copies L_1 and L_{-1}, where addition is defined componentwise, but multiplication is given by

$$(k, \ell) \cdot (k', \ell') = (kk' + \ell\ell', k\ell' + \ell k').$$

In other words, $D(L)$ is multiplicatively graded by $\{\pm 1\}$.

$D(L)$ is endowed with the product partial order, i.e., $(k', \ell') \geq (k, \ell)$ when $k' \geq k$ and $\ell' \geq \ell$.

Here is an example relating to "patchworking," [ItMS].

Example 7.5. *Suppose $\mathcal{G}$ is an ordered Abelian monoid, viewed as a semiring† as in Construction 7.1. Define the* ***doubled layered domain†***

$$R = \mathscr{R}(D(L), \mathcal{G}) = \{((k, \ell), a) : (k, \ell) \neq (0, 0),\ a \in \mathcal{G}\},$$

but with addition and multiplication given by the following rules:

$$((k,\ell),a) + ((k',\ell'),b) = \begin{cases} ((k,\ell),a) & \text{if } a > b, \\ ((k',\ell'),b) & \text{if } a < b, \\ ((k+k', \ell+\ell'),\ a) & \text{if } a = b. \end{cases}$$

$$((k,\ell),a) \cdot ((k',\ell'),b) = ((kk' + \ell\ell', k\ell' + k'\ell),\ ab).$$

Remark 7.6. *In $R = \mathscr{R}(D(L), \mathcal{G})$, the symmetry $\sigma : R \to R$ given by $\sigma : ((k, \ell), a) \mapsto ((\ell, k), a)$ is analogous to the one described in* [AkGG]*, and behaves much like negation.*

For example, when $L = \{1, \infty\}$, we note that $D(L) = \{(1, 1), (1, \infty), (\infty, 1), (\infty, \infty)\}$, which is applicable to Viro's theory of patchworking, where the "tangible" part could be viewed as those elements of layer $(1, 1), (1, \infty)$, or $(\infty, 1)$. Explicitly, comparing with Viro's use of hyperfields in [Vi, § 3.5], we identify these three layers respectively with $0, 1$, and -1 in his terminology, and the element (∞, ∞) with the set $\{0, 1, -1\}$.

8. Matrices and linear algebra

As an application, the supertropical and layered structures provide many of the analogs to the classical Hamilton-Cayley-Frobenius theory. $M_n(R)$ denotes the semiring† of $n \times n$ matrices over a semiring R. (Note that the familiar matrix operations do not require negation.)

Although one of the more popular and most applicable aspects of idempotent mathematics, idempotent matrix theory is handicapped by the lack of an element -1 with which to construct the determinant. Many ingenious methods have been devised to circumvent this difficulty, as surveyed in [AkBG]; also cf. [AkGG] and many interesting papers in this volume. Unfortunately these gives rise to many different notions of rank of matrix, and often are difficult to understand. The layered (and more specifically, supertropical) theories give a unified and relatively straightforward notion of rank of a matrix, eigenvalue, adjoint, etc.

8.1. **The supertropical determinant.** This discussion summarizes [IzR3]. We define the **supertropical determinant** $|A|$ of a matrix $A = (a_{i,j})$ to be the permanent:

$$|(a_{i,j})| = \sum_{\pi \in S_n} a_{1,\pi(1)} \cdots a_{n,\pi(n)}. \tag{8.1}$$

Defining the **transpose matrix** $(a_{i,j})^{\mathrm{t}}$ to be $(a_{j,i})$, we have

$$\left|(a_{i,j})^{\mathrm{t}}\right| = |(a_{i,j})|.$$

$|A| = \mathbb{0}_R$ iff "enough" entries are $\mathbb{0}_R$ to force each summand in Formula 8.1 to be $\mathbb{0}_R$. This property, which in classical matrix theory provides a description of singular subspaces, is too strong for our purposes. We now take the natural supertropical version. Write $\mathcal{T}$ for the tangible elements of our supertropical semiring R, and $\mathcal{T}_{\mathbb{0}} = \mathcal{T} \cup \{\mathbb{0}\}$.

Definition 8.1. *A matrix A is **nonsingular** if $|A| \in \mathcal{T}$; A is **singular** when $|A| \in \mathcal{G}_{\mathbb{0}}$.*

The standard supertropical structure often is sufficient for matrices, since it enables us to distinguish between nonsingular matrices (in which the tropical $n \times n$ determinant is computed as the unique maximal product of n elements in one track) and singular matrices.

The tropical determinant is not multiplicative, as seen by taking the nonsingular matrix $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is singular and $\left|A^2\right| = 5^\nu \neq 2 \cdot 2$. But we do have:

Theorem 8.2. *For any $n \times n$ matrices over a supertropical semiring R, we have*

$$|AB| \underset{\text{gs}}{\models} |A|\,|B|\,.$$

In particular, $|AB| = |A|\ |B|$ whenever $|AB|$ is tangible.

We say a permutation $\sigma \in S_n$ **attains** $|A|$ if $|A| \cong_\nu a_{\sigma(1),1} \cdots a_{\sigma(n),n}$.

- By definition, some permutation always attains $|A|$.
- If there is a unique permutation σ which attains $|A|$, then $|A| = a_{1,\sigma(1)} \ \cdots \ a_{n,\sigma(n)}$.
- If at least two permutations attain $|A|$, then A must be singular. Note in this case that if we replaced all nonzero entries of A by tangible entries of the same ν-value, then A would still be singular.

8.2. Quasi-identities and the adjoint.

Definition 8.3. *A **quasi-identity** matrix $I_{\mathcal{G}}$ is a nonsingular, multiplicatively idempotent matrix equal to $I + Z_{\mathcal{G}}$, where $Z_{\mathcal{G}}$ is $\mathbb{0}_R$ on the diagonal, and whose off-diagonal entries are ghosts or $\mathbb{0}_R$.*

$|I_{\mathcal{G}}| = \mathbb{1}_R$ by the nonsingularity of $I_{\mathcal{G}}$. Also, for any matrix A and any quasi-identity, $I_{\mathcal{G}}$, we have $AI_{\mathcal{G}} = A + A_{\mathcal{G}}$, where $A_{\mathcal{G}} = AZ_{\mathcal{G}} \in M_n(\mathcal{G}_{\mathbb{0}})$.

There is another notion to help us out.

Definition 8.4. *The (i,j)-**minor** $A'_{i,j}$ of a matrix $A = (a_{i,j})$ is obtained by deleting the i row and j column of A. The **adjoint** matrix $\operatorname{adj}(A)$ of A is defined as the transpose of the matrix $(a'_{i,j})$, where $a'_{i,j} = \left|A'_{i,j}\right|$.*

Remark 8.5.

(i) *Suppose $A = (a_{i,j})$. An easy calculation using Formula (8.1) yields*

$$|A| = \sum_{j=1}^{n} a_{i,j}\, a'_{i,j}, \quad \forall i. \tag{8.2}$$

Consequently, $a_{i,j}\, {a'}_{i,j} \leq_\nu |A|$ for each i, j.

(ii) *If we take $k \neq i$, then replacing the i row by the k row in A yields a matrix with two identical rows; thus, its determinant is a ghost, and we thereby obtain*

$$\sum_{j=1}^{n} a_{i,j}\, a'_{k,j} \in \mathcal{G}_{\mathbb{0}}, \qquad \forall k \neq i; \tag{8.3}$$

Likewise

$$\sum_{j=1}^{n} a_{j,i}\, a'_{j,k} \in \mathcal{G}_{\mathbb{0}}, \qquad \forall k \neq i.$$

One easily checks that $\operatorname{adj}(B)\operatorname{adj}(A) = \operatorname{adj}(AB)$ for any 2×2 matrices A and B. However, this fails for larger n, cf. [IzR3, Example 4.7]. We do have the following fact, which illustrates the subtleties of the supertropical structure, cf. [IzR3, Proposition 5.6]:

Proposition 8.6. $\operatorname{adj}(AB) = \operatorname{adj}(B)\operatorname{adj}(A) + ghost$.

Definition 8.7. *For $|A|$ invertible, define*

$$I_A = A\frac{\operatorname{adj}(A)}{|A|}, \qquad I'_A = \frac{\operatorname{adj}(A)}{|A|}A.$$

The matrices I_A and I'_A are quasi-identities, as seen in [IzR3, Theorem 4.13]. The main technique of proof is to define a **string** (from the matrix A) to be a product $a_{i_1,j_1}\cdots a_{i_k,j_k}$ of entries from A and, given such a string, to define its **digraph** to be the graph whose edges are $(i_1,j_1),\dots,(i_k,j_k)$, counting multiplicities. An k-**multicycle** in a digraph is the union of disjoint simple cycles, the sum of whose lengths is k; thus every vertex in an n-multicycle appears exactly once. A careful examination of the digraph in conjunction with Hall's Marriage Theorem yields the following major results from [IzR3, Theorem 4.9 and Theorem 4.12]:

Theorem 8.8.
(i) $|A\operatorname{adj}(A)| = |A|^n$.
(ii) $|\operatorname{adj}(A)| = |A|^{n-1}$.

In case A is a nonsingular, we define

$$A^\nabla = \frac{\operatorname{adj}(A)}{|A|}.$$

Thus $AA^\nabla = I_A$, and $A^\nabla A = I'_A$. Note that I'_A and I_A may differ off the diagonal, although

$$I_A A = AA^\nabla A = AI'_A.$$

This result is refined in [IzR4, Theorem 2.18]. One might hope that $A\operatorname{adj}(A)A = |A|\,A$, but this is false in general! The difficulty is that one might not be able to extract an n-multicycle from

$$a_{i,j}a'_{k,j}a_{k,\ell}. \tag{8.4}$$

For example, when $n = 3$, the term $a_{1,1}(a_{1,3}a_{3,2})a_{2,2} = a_{1,1}a'_{2,1}a_{2,2}$ does not contain an n-multicycle. We do have the following positive result from [IzR4, Theorem 4.18]:

Theorem 8.9. $\operatorname{adj}(A)\operatorname{adj}(\operatorname{adj}(A))\operatorname{adj}(A) \cong_\nu |A|^{n-1}\operatorname{adj}(A)$ *for any $n\times n$ matrix A.*

8.3. **The supertropical Hamilton-Cayley theorem.**

Definition 8.10. *Define the **characteristic polynomial** f_A of the matrix A to be*

$$f_A = |\lambda I + A|,$$

*and the **tangible characteristic polynomial** to be a tangible polynomial $\hat f_A = \lambda^n + \sum_{i=1}^n \hat\alpha_i\lambda^{n-i}$, where $\hat\alpha_i$ are tangible and $\hat\alpha_I \cong_\nu \alpha_i$, such that $f_A = \lambda^n + \sum_{i=1}^n \alpha_i\lambda^{n-i}$.*

Under this notation, we see that $\alpha_k \in R$ arises from the dominant k-multicycles in the digraph of A. We say that a matrix A **satisfies** a polynomial $f \in R[\lambda]$ if $f(A) \in M_n(\mathcal{G}_{\mathbb{0}})$.

Theorem 8.11. ***(Supertropical Hamilton-Cayley)*** *Any matrix A satisfies both its characteristic polynomial f_A and its tangible characteristic polynomial $\hat f_A$.*

8.4. **Tropical dependence.** Now we apply supertropical matrix theory to vectors. As in classical mathematics, one defines a **module** (often called **semi-module** in the literature) analogously to module in classical algebra, noting again that negation does not appear in the definition. It is convenient to stipulate that the module V has a zero element $\mathbb{0}_V$, and then we need the axiom:

$$a\mathbb{0}_V = \mathbb{0}_V \quad \text{for all} \quad a \in R.$$

Also, if $\mathbb{0} \in R$ then we require that $\mathbb{0}v = \mathbb{0}_V$ for all $v \in V$.

In what follows, F always denotes a 1-semifield. In this case, a module over F is called a (supertropical) **vector space**. The natural example is $F^{(n)}$, with componentwise operations. As in the classical theory, there is the usual familiar correspondence between the semiring $M_n(F)$ and the linear transformations of $F^{(n)}$.

For $v = (v_1,\dots,v_n), w = (w_1,\dots,w_n) \in F^{(n)}$, we write $v \underset{\text{gs}}{\models} w$ when $v_i \underset{\text{gs}}{\models} w_i$ for all $1 \le i \le n$.

Here is an application of the adjoint matrix, used to solve equations.

Remark 8.12. *Suppose A is nonsingular, and $v \in F^{(n)}$. Then the equation $Aw = v +$ ghost has the solution $w = A^{\nabla} v$. Indeed, writing $I_A = I + Z_{\mathcal{G}}$ for matrix $Z_{\mathcal{G}}$ ghost, we have*

$$Aw = AA^{\nabla} v = I_A v = (I + Z_{\mathcal{G}})v \underset{\text{gs}}{\models} v.$$

This leads to the supertropical analog of Cramer's rule [IzR4, Theorem 3.5]:

Theorem 8.13. *If A is a nonsingular matrix and v is a tangible vector, then the equation $Ax \underset{\text{gs}}{\models} v$ has a solution over F which is the tangible vector having value $A^{\nabla} v$.*

Our next task is to characterize singularity of a matrix A in terms of "tropical dependence" of its rows. In some ways the standard supertropical theory works well with matrices, since we are interested mainly in whether or not this matrix is nonsingular, i.e., if its determinant is tangible; at the outset, at least, we are not concerned with the precise ghost layer of the determinant.

Definition 8.14. *A subset $W \subset F^{(n)}$ is* ***tropically dependent*** *if there is a finite sum $\sum \alpha_i w_i \in \mathcal{G}_{\mathbb{0}}^{(n)}$, with each $\alpha_i \in \mathcal{T}_{\mathbb{0}}$, but not all of them $\mathbb{0}_R$; otherwise $W \subset F^{(n)}$ is called* ***tropically independent****. A vector $v \in F^{(n)}$ is* ***tropically dependent*** *on W if $W \cup \{v\}$ is tropically dependent.*

By [IzKR2, Proposition 4.5], we have:

Proposition 8.15. *Any $n + 1$ vectors in $F^{(n)}$ are tropically dependent.*

Theorem 8.16. *(*[IzR3, Theorem 6.5]*) Vectors $v_1, \dots, v_n \in F^{(n)}$ are tropically dependent, iff the matrix whose rows are $v_1, \dots, v_n$ is singular.*

Corollary 8.17. *The matrix $A \in M_n(F)$ over a supertropical domain F is nonsingular iff the rows of A are tropically independent, iff the columns of A are tropically independent.*

Proof. Apply the theorem to $|A|$ and $|A^t|$, which are the same. □

There are two competing supertropical notions of base of a vector space, that of a maximal independent set of vectors, and that of a minimal spanning set, but this is unavoidable since, unlike the classical theory, these two definitions need not coincide.

8.5. **Supertropical eigenvectors.** The standard definition of an **eigenvector** of a matrix A is a vector v, with **eigenvalue** β, satisfying $Av = \beta v$. It is well known [BrR] that any (tangible) matrix has an eigenvector.

Example 8.18. *The characteristic polynomial f_A of*

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

is $(\lambda + 4)(\lambda + 1) + 0 = (\lambda + 4)(\lambda + 1)$, and the vector $(4, 0)$ is a eigenvector of A, with eigenvalue 4. However, there is no eigenvector having eigenvalue 1.

In general, the lesser roots of the characteristic polynomial are "lost" as eigenvalues. We rectify this deficiency by weakening the standard definition.

Definition 8.19. *A tangible vector v is a* ***generalized supertropical eigenvector*** *of a (not necessarily tangible) matrix A, with* ***generalized supertropical eigenvalue*** *$\beta \in \mathcal{T}_{\mathbb{0}}$, if $A^m v \underset{\text{gs}}{\models} \beta^m v$ for some m; the minimal such m is called the* ***multiplicity*** *of the eigenvalue (and also of the eigenvector). A* ***supertropical eigenvector*** *is a generalized supertropical eigenvector of multiplicity 1.*

Example 8.20. *The matrix $A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ of Example 8.18 also has the tangible supertropical eigenvector $v = (0, 4)$, corresponding to the supertropical eigenvalue 1, since*

$$Av = (4^{\nu}, 5) = 1v + (4^{\nu}, -\infty).$$

Proposition 8.21. *If v is a tangible supertropical eigenvector of A with supertropical eigenvalue β, the matrix $A + \beta I$ is singular (and thus β must be a (tropical) root of the characteristic polynomial f_A of A).*

Conversely, we have:

Theorem 8.22 ([IzR3, Theorem 7.10]). *Assume that $\nu|_{\mathcal{T}} : \mathcal{T} \to \mathcal{G}$ is 1:1. For any matrix A, the dominant tangible root of the characteristic polynomial of A is an eigenvalue of A, and has a tangible eigenvector. The other tangible roots are precisely the supertropical eigenvalues of A.*

Let us return to our example $A = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$. Its characteristic polynomial is $\lambda^2 + 2\lambda + 2 = (\lambda + 0)(\lambda + 2)$, whose roots are 2 and 0. The eigenvalue 2 has tangible eigenvector $v = (0, 2)$ since $Av = (2, 4) = 2v$, but there are no other tangible eigenvalues. A does have the tangible supertropical eigenvalue 0, with tangible supertropical eigenvector $w = (2, 1)$, since $Aw = (2, 3^\nu) = 0w + (-\infty, 3^\nu)$. Note that $A + 0I = \begin{pmatrix} 0^\nu & 0 \\ 1 & 2 \end{pmatrix}$ is singular; i.e., $|A + 0I| = 2^\nu$.

Furthermore, $A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a root of $\lambda^2 + 4A$, and thus A is a root of $g = \lambda^4 + 4\lambda^2 = (\lambda(\lambda + 2))^2$, but 0 is not a root of g although it is a root of f_A. This shows that the naive formulation of Frobenius' theorem fails in the supertropical theory, and is explained in the work of Adi Niv [Niv].

8.6. **Tropical bases and rank.** The customary definition of tropical base, which we call **s-base** (for **spanning base**), is a minimal spanning set (when it exists). However, this definition is rather restrictive, and a competing notion provides a richer theory.

Definition 8.23. *A **d-base** (for **dependence base**) of a vector space V is a maximal set of tropically independent elements of V. A **d,s-base** is a d-base which is also an s-base. The **rank** of a set $\mathcal{B} \subseteq V$, denoted* $\operatorname{rank}(\mathcal{B})$*, is the maximal number of d-independent vectors of $\mathcal{B}$.*

Our d-base corresponds to the "basis" in [MS, Definition 5.2.4]. In view of Proposition 8.15, all d-bases of $F^{(n)}$ have precisely n elements.

This leads us to the following definition.

Definition 8.24. *The **rank** of a vector space V is defined as:*

$$\operatorname{rank}(V) := \max \left\{ \operatorname{rank}(\mathcal{B}) : \mathcal{B} \text{ is a d-base of } V \right\}.$$

We have just seen that $\operatorname{rank}(F^{(n)}) = n$. Thus, if $V \subset F^{(n)}$, then $\operatorname{rank}(V) \leq n$.

We might have liked $\operatorname{rank}(V)$ to be independent of the choice of d-base of V, for any vector space V. This is proved in the classical theory of vector spaces by showing that dependence is transitive. However, transitivity of dependence fails in the supertropical theory, and, in fact, different d-bases may contain different numbers of elements, even when tangible. An example is given in [MS, Example 5.4.20], and reproduced in [IzKR2, Example 4.9] as being a subspace of $F^{(4)}$ having d-bases both of ranks 2 and 3.

Example 8.25. *The matrix $A = \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 1 \\ 4 & 4 & 2 \end{pmatrix}$ has rank 2, but is "ghost annihilated" by the tropically independent vectors $v_1 = (1, 1, 0)^{\mathrm{t}}$ and $v_2 = (1, 1, 1)^{\mathrm{t}}$; i.e., $Av_1 = Av_2 = (5^\nu, 5^\nu, 5^\nu)^{\mathrm{t}}$, although $2 + 2 > 3$.*

We do have some consolations.

Proposition 8.26 ([IzKR2, Proposition 4.11]). *For any tropical subspace V of $F^{(n)}$ and any tangible $v \in V$, there is a tangible d-base of V containing v whose rank is that of V.*

Proposition 8.27 ([IzKR2, Proposition 4.13]). *Any $n \times n$ matrix of rank m has ghost annihilator of rank $\geq n - m$.*

8.6.1. *Semi-additivity of rank.*

Definition 8.28. *A function* $\operatorname{rank}_{\mathrm{S}} : S \to \mathbb{N}$ *is **monotone** if for all $S_2 \subseteq S_1 \subseteq S$ we have*

$$\operatorname{rank}_{\mathrm{S}}(S_2 \cup \{s\}) - \operatorname{rank}_{\mathrm{S}}(S_2) \geq \operatorname{rank}_{\mathrm{S}}(S_1 \cup \{s\}) - \operatorname{rank}_{\mathrm{S}}(S_1) \tag{8.5}$$

for all $s \in S$.

Note that (8.5) says that $\operatorname{rank}_{\mathrm{S}}(S_1) - \operatorname{rank}_{\mathrm{S}}(S_2) \geq \operatorname{rank}_{\mathrm{S}}(S_1 \cup \{s\}) - \operatorname{rank}_{\mathrm{S}}(S_2 \cup \{s\})$. Also, taking $S_2 = \emptyset$ yields $\operatorname{rank}_{\mathrm{S}}(S_1 \cup \{s\}) - \operatorname{rank}_{\mathrm{S}}(S_1) \leq 1$.

Lemma 8.29. *If* $\mathrm{rank_S} : S \to \mathbb{N}$ *is monotone, then*

$$\mathrm{rank_S}(S_1) + \mathrm{rank_S}(S_2) \geq \mathrm{rank_S}(S_1 \cup S_2) + \mathrm{rank_S}(S_1 \cap S_2) \tag{8.6}$$

for all $S_1, S_2 \subset S$.

Proof. Induction on $m = \mathrm{rank_S}(S_2 \setminus S_1)$. If $m = 0$, i.e., $S_2 \subseteq S_1$, then the left side of (8.6) equals the right side. Thus we may assume that $m \geq 1$. Pick s in a d-base of $S_2 \setminus S_1$. Let $S_2' = S_2 \setminus \{s\}$. Noting that $\mathrm{rank_S}(S_2' \setminus S_1) = m - 1$, we see by induction that

$$\mathrm{rank_S}(S_1) + \mathrm{rank_S}(S_2') \geq \mathrm{rank_S}(S_1 \cup S_2') + \mathrm{rank_S}(S_1 \cap S_2'), \tag{8.7}$$

or (taking $S_1 \cup S_2'$ instead of S_2 in (8.5)),

$$\begin{aligned} \mathrm{rank_S}(S_1) - \mathrm{rank_S}(S_1 \cap S_2) = \mathrm{rank_S}(S_1) - \mathrm{rank_S}(S_1 \cap S_2') &\geq \mathrm{rank_S}(S_1 \cup S_2') - \mathrm{rank_S}(S_2') \\ &\geq \mathrm{rank_S}(S_1 \cup S_2) - \mathrm{rank_S}(S_2), \end{aligned}$$

yielding (8.6). □

Proposition 8.30. $\mathrm{rank}(S_1) + \mathrm{rank}(S_2) \geq \mathrm{rank}(S_1 \cup S_2) + \mathrm{rank}(S_1 \cap S_2)$ *for all* $S_1, S_2 \subset S$.

Proof. rank is a monotone function, since each side of (8.5) is 0 or 1, depending on whether or not s is independent of S_i, and only decreases as we enlarge the set. □

8.7. **Bilinear forms and orthogonality.** One can refine the study of bases by introducing angles, i.e., orthogonality, in terms of bilinear forms. Let us quote some results from [IzKR2].

Definition 8.31. *A* ***(supertropical) bilinear form*** B *on a (supertropical) vector space* V *is a function* $B : V \times V \to F$ *satisfying*

$$B(v_1 + v_2, w_1 + w_2) \underset{\mathrm{gs}}{\models} B(v_1, w_1) + B(v_1, w_2) + B(v_2, w_1) + B(v_2, w_2),$$

$$B(\alpha v_1, w_1) = \alpha B(v_1, w_1) = B(v_1, \alpha w_1),$$

for all $\alpha \in F$ *and* $v_i \in V$, *and* $w_j \in V'$.

We work with a fixed bilinear form $B = \langle\ ,\ \rangle$ on a (supertropical) vector space $V \subseteq F^{(n)}$. The **Gram matrix** of vectors $v_1, \dots, v_k \in F^{(n)}$ is defined as the $k \times k$ matrix

$$\widetilde{G}(v_1, \dots, v_k) = \begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_2 \rangle & \cdots & \langle v_1, v_k \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle & \cdots & \langle v_2, v_k \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_k, v_1 \rangle & \langle v_k, v_2 \rangle & \cdots & \langle v_k, v_k \rangle \end{pmatrix}. \tag{8.8}$$

The set $\{v_1, \dots, v_k\}$ is **nonsingular** (with respect to B) when its Gram matrix is nonsingular.

In particular, given a vector space V with s-base $\{b_1 \dots, b_k, \}$ we have the matrix $\widetilde{G} = \widetilde{G}(b_1 \dots, b_k)$, which can be written as $(g_{i,j})$ where $g_{i,j} = \langle b_i, b_j \rangle$. The singularity of $\widetilde{G}$ does not depend on the choice of s-base.

Definition 8.32. *For vectors* v, w *in* V, *we write* $v \perp\!\!\perp w$ *when* $\langle v, w \rangle \in \mathcal{G}_{\mathbb{0}}$, *that is* $\langle v, w \rangle \underset{\mathrm{gs}}{\models} \mathbb{0}_F$, *and say that* v *is* ***left ghost orthogonal*** *to* w. *We write* $W^{\perp\!\!\perp}$ *for* $\{v \in V : v \perp\!\!\perp w \text{ for all } w \in W.\}$

Definition 8.33. *A subspace* W *of* V *is called* ***nondegenerate*** *(with respect to* B*), if* $W^{\perp\!\!\perp} \cap W$ *is ghost. The bilinear form* B *is* ***nondegenerate*** *if the space* V *is nondegenerate.*

Lemma 8.34. *Suppose* $\{w_1, \dots, w_m\}$ *tropically spans a subspace* W *of* V, *and* $v \in V$. *If* $\sum_{i=1}^m \beta_i \langle v, w_i \rangle \in \mathcal{G}_{\mathbb{0}}$ *for all* $\beta_i \in \mathcal{T}$, *then* $v \in W^{\perp\!\!\perp}$.

Theorem 8.35. *(*[IzKR2, Theorem 6.7]*) Assume that vectors* $w_1 \dots, w_k \in V$ *span a nondegenerate subspace* W *of* V. *If* $|\widetilde{G}(w_1 \dots, w_k)| \in \mathcal{G}_{\mathbb{0}}$, *then* $w_1 \dots, w_k$ *are tropically dependent.*

Corollary 8.36. *If the bilinear form* B *is nondegenerate on a vector space* V, *then the Gram matrix (with respect to any given supertropical d,s-base of* V*) is nonsingular.*

Definition 8.37. *The bilinear form* B *is* ***supertropically alternate*** *if* $\langle v, v \rangle \in \mathcal{G}_{\mathbb{0}}$ *for all* $v \in V$. B *is* ***supertropically symmetric*** *if* $\langle v, w \rangle + \langle w, v \rangle \in \mathcal{G}_{\mathbb{0}}$ *for all* $v, w \in V$.

We aim for the supertropical version ([IzKR2, Theorem 6.19]) of a classical theorem of Artin, that any bilinear form in which ghost-orthogonality is symmetric must be a supertropically symmetric bilinear form.

Definition 8.38. *The (supertropical) bilinear form B is* ***orthogonal-symmetric*** *if it satisfies the following property for for any finite sum, with $v_i, w \in V$:*

$$\sum_i \langle v_i, w\rangle \in \mathcal{G}_{\mathbb{0}} \quad \textit{iff} \quad \sum_i \langle w, v_i\rangle \in \mathcal{G}_{\mathbb{0}}, \tag{8.9}$$

B is ***supertropically orthogonal-symmetric*** *if B is orthogonal-symmetric and satisfies the additional property that $\langle v, w\rangle \cong_\nu \langle w, v\rangle$ for all $v, w \in V$ satisfying $\langle v, w\rangle \in \mathcal{T}$.*

The symmetry condition extends to sums, and after some easy lemmas we obtain ([IzKR2, Theorem 6.19]):

Theorem 8.39. *Every orthogonal-symmetric bilinear form B on a vector space V is supertropically symmetric.*

9. Identities of semirings, especially matrices

The word "identity" has several interpretations, according to its context. First of all, there are well-known matrix identities such as the Hamilton-Cayley identity which says that any matrix is a root of its characteristic polynomial.

Since the classical theory of polynomial identities is tied in with invariant theory, we also introduce layered polynomial identities (PIs), to enrich our knowledge of layered matrices.

9.1. Polynomial identities of semirings†.

We draw on basic concepts of polynomial identities, i.e., PI's, say from [R2, Chapter 23]. Since semirings† do not involve negatives, we modify the definition a bit.

Definition 9.1. *The* ***free*** *$\mathbb{N}$-**semiring**† $\mathbb{N}\{x_1, x_2, \dots\}$ is the monoid semiring† of the free (word) monoid $\{x_1, x_2, \dots\}$ over the commutative semiring† $\mathbb{N}$.*

Definition 9.2. *A* ***(semiring†) polynomial identity*** *(PI) of a semiring† R is a pair (f, g) of (noncommutative) polynomials $f(x_1, \dots, x_m), g(x_1, \dots, x_m) \in \mathbb{N}\{x_1, \dots x_m\}$ for which*

$$f(r_1, \dots, r_m) = g(r_1, \dots, r_m), \quad \forall r_1, \dots, r_m \in R.$$

We write $(f, g) \in \mathrm{id}(R)$ when (f, g) is a PI of R.

Remark 9.3. *Analogously, one defines a* ***semigroup identity*** *of a semigroup $\mathcal{S}$ to be a pair (f, g) of (noncommutative) monomials $f(x_1, \dots, x_m), g(x_1, \dots, x_m) \in \mathbb{N}\{x_1, \dots x_m\}$ for which $f(s_1, \dots, s_m) = g(s_1, \dots, s_m)$, $\forall s_1, \dots, s_m \in \mathcal{S}$. If $\mathcal{S}$ is contained in the multiplicative semigroup of a semiring† R, the semigroup identities of $\mathcal{S}$ are precisely the semiring† PIs (f, g) where f and g are monomials.*

Akian, Gaubert and Guterman [AkGG, Theorem 4.21] proved their **strong transfer principle**, which immediately implies the following easy but important observation:

Theorem 9.4. *If $f, g \in \mathbb{N}\{x_1, \dots, x_n\}$ have disjoint supports and $f - g$ is a PI of $M_n(\mathbb{Z})$, then $f = g$ is also a semiring† PI of $M_n(R)$ for any commutative semiring† R.*

Proof. Since $\mathbb{Z}$ is an infinite integral domain, $f - g$ is also a PI of $M_n(C)$, where $C = \mathbb{Z}[\xi_1, \xi_2, \dots]$ denotes the free commutative ring in countably many indeterminates, implying (f, g) is a semiring† PI of $M_n(\mathbb{N}[\xi_1, \xi_2, \dots])$. But the semiring† $M_n(R)$ is a homomorphic image of $M_n(\mathbb{N}[\xi_1, \xi_2, \dots])$, implying $(f, g) \in \mathrm{id}(M_n(R))$. □

Corollary 9.5. *Any PI of $M_n(\mathbb{Z})$ yields a corresponding semiring† PI of $M_n(R)$ for all commutative semirings† R.*

Proof. Take f to be the sum of the terms having positive coefficient, and g to be the sum of the terms having negative coefficient, and apply the theorem. □

Many (but not all) matrix PIs can be viewed in terms of Theorem 9.4, although semiring versions of basic results such as the Amitsur-Levitzki Theorem and Newton's Formulas often are more transparent here.

We say that polynomials $f(x_1, \dots, x_m)$ and $g(x_1, \dots, x_m)$ are a ***t*-alternating pair** if f and g are interchanged whenever we interchange a pair x_i and x_j for some $1 \le i < j \le t$. For example, x_1x_2 and x_2x_1 are a 2-alternating pair. Sometimes we write the non-alternating variables as $y_1, y_2, \dots$; we write y as shorthand for all the y_j.

Definition 9.6. *We partition the symmetric group S_t of permutations in n letters into the even permutations S_t^+ and the odd permutations S_t^-. Given a t-linear polynomial $h(x_1,\dots,x_t;y)$, we define the t-**alternating pair***

$$h^+_{\mathrm{alt}}(x_1,\dots,x_t;y) := \sum_{\sigma\in S_t^+} h(x_{\sigma(1)},\dots,x_{\sigma(t)};y)$$

and

$$h^-_{\mathrm{alt}}(x_1,\dots,x_t;y) := \sum_{\sigma\in S_t^-} h(x_{\sigma(1)},\dots,x_{\sigma(t)};y).$$

*The **standard pair** is* $\mathrm{Stn}_t := (h^+_{\mathrm{alt}}, h^-_{\mathrm{alt}})$, *where $h = x_1\cdots x_t$. Explicitly,*

$$\mathrm{Stn}_t := \left(\sum_{\sigma\in S_t^+} x_{\sigma(1)}\cdots x_{\sigma(t)}, \sum_{\sigma\in S_t^-} x_{\sigma(1)}\cdots x_{\sigma(t)}\right).$$

*The **Capelli pair** is* $\mathrm{Cap}_t := (h^+_{\mathrm{alt}}, h^-_{\mathrm{alt}})$, *where $h = x_1y_1x_2y_2\cdots x_ty_t$. Explicitly,*

$$\mathrm{Cap}_t := \left(\sum_{\sigma\in S_t^+} x_{\sigma(1)}y_1x_{\sigma(2)}y_2\cdots y_{t-1}x_{\sigma(t)}y_t, \sum_{\sigma\in S_t^-} x_{\sigma(1)}y_1x_{\sigma(2)}y_2\cdots y_{t-1}x_{\sigma(t)}y_t\right).$$

Proposition 9.7. *Any t-alternating pair (f,g) is a PI for every semiring† R spanned by fewer than t elements over its center.*

Proof. Suppose R is spanned by $\{b_1,b_2,\dots,b_{t-1}\}$. We need to verify

$$f\left(\sum \alpha_{i,1}b_{i_1},\dots,\sum\alpha_{i,t}b_{i_t},\dots\right) = g\left(\sum \alpha_{i,1}b_{i_1},\dots,\sum\alpha_{i,t}b_{i_t},\dots\right).$$

Since f and g are linear in these entries, it suffices to verify

$$f(b_{i_1},\dots,b_{i_t},\dots) = g(b_{i_1},\dots,b_{i_t},\dots) \tag{9.1}$$

for all $i_1,\dots,i_t$. But by hypothesis, two of these must be equal, say i_k and $i_{k'}$, so switching these two yields (9.1) by the alternating hypothesis. □

Let $e_{i,j}$ denote the matrix units. The semiring† version of the Amitsur-Levitzki theorem [AmL], that $\mathrm{Stn}_{2n} \in \mathrm{id}(M_n(\mathbb{N}))$, is an immediate consequence of Theorem 9.4, and its minimality follows from:

Lemma 9.8. *Any pair of multilinear polynomials $f(x_1,\dots,x_m)$ and $g(x_1,\dots,x_m)$ having no common monomials do not comprise a PI of $M_n(R)$ unless $m \ge 2n$.*

Proof. Rewriting indices we may assume that $x_1\cdots x_m$ appears as a monomial of f, but not of g, and we note (for $\ell = \left[\frac{m}{2}\right]+1$) that

$$\begin{aligned} f(e_{1,1},e_{1,2},e_{2,2},e_{2,3},\dots,e_{k-1,k},e_{k,k},\dots) &= e_{1,\ell} \quad \neq 0;\\ g(e_{1,1},e_{1,2},e_{2,2},e_{2,3},\dots,e_{k-1,k},e_{k,k},\dots) &= 0 \end{aligned}$$

where $m = \left[\frac{d}{2}\right]$, a contradiction. □

Likewise, the identical proof of [R2, Remark 23.14] shows that the Capelli pair Cap_{n^2} is not a PI of $M_n(C)$, and in fact $(e_{1,1},0)\in \mathrm{Cap}_{n^2}(M_n(R))$ for any semiring† R.

9.2. **Surpassing identities.** The **surpassing identity** $f \underset{\mathrm{gs}}{\models} g$ holds when $f(a_1,\dots,a_m) \underset{\mathrm{gs}}{\models} g(a_1,\dots,a_m)$ for all $a_1,\dots,a_n\in R$.

Example 9.9. *Take the general 2×2 matrix $A = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$. Then $\mathrm{tr}(A) = a+d$ and $|A| = ad+bc$. $A^2 = \begin{pmatrix} a^2+bc & b(a+d)\\ c(a+d) & bc+d^2\end{pmatrix}$, so*

$$A^2 + adI = \begin{pmatrix} a(a+d)+bc & b(a+d)\\ c(a+d) & bc+d(a+d)\end{pmatrix} = \mathrm{tr}(A)A + bcI,$$

implying

$$A^2 + |A|\,I = \mathrm{tr}(A)A + bc^\nu I,$$

yielding the surpassing identity $A^2 + |A|\, I \underset{\mathrm{gs}}{\models} \mathrm{tr}(A)A$ *for* 2×2 *matrices.*

We might hope for a surpassing identity involving alternating terms in the Hamilton-Cayley polynomial, but a cursory examination of matrix cycles dashes our hopes.

Example 9.10. *Let* $A = \begin{pmatrix} - & d & a \\ c & - & - \\ - & b & - \end{pmatrix}$. *Then* $A^2 = \begin{pmatrix} cd & ab & - \\ - & cd & ac \\ bc & - & - \end{pmatrix}$ *and* $A^3 = \begin{pmatrix} abc & cd^2 & acd \\ c^2d & abc & - \\ - & bcd & abc \end{pmatrix}$, *implying*

$$A^3 = \alpha A + |A|$$

in this case, where α *denotes the other coefficient in* f_A. *But for* $A = \begin{pmatrix} a & - & - \\ - & b & - \\ - & - & c \end{pmatrix}$ *we have*

$$A^3 + \alpha A + 2 \begin{pmatrix} - & - & - \\ - & abc & - \\ - & - & abc \end{pmatrix} = \mathrm{tr}(A)A^2 + |A|,$$

so neither $A^3 + \alpha A$ *nor* $\mathrm{tr}(A)A^2 + |A|$ *necessarily surpasses the other.*

9.3. **Layered surpassing identities.** Since we want to deal with general layers, we write $2a$ (instead of a^ν) for $a + a$, but note that $s(2a) = 2s(a)$. When working with the layered structure, we can extend the notion of PI from Definition 9.2 by making use of the following relations that arise naturally in the theory.

Definition 9.11. *The* L***-surpassing relation*** $\underset{L}{\models}$ *is given by*

$$a \underset{L}{\models} b \quad \text{iff either} \quad \begin{cases} a = b + c & \text{with} \quad c \quad s(b)\text{-ghost}, \\ a = b, & \\ a \cong_\nu b & \text{with} \quad a \quad s(b)\text{-ghost}. \end{cases} \tag{9.2}$$

It follows that if $a \underset{L}{\models} b$, then $a + b$ is $s(b)$-ghost. When $a \neq b$, this means $a \geq_\nu b$ and a is $s(b)$-ghost.

Definition 9.12. *The* ***surpassing*** (L, ν)***-relation*** $\underset{L}{\models}_\nu$ *is given by*

$$a \underset{L}{\models}_\nu b \qquad \text{iff} \qquad a \underset{L}{\models} b \quad \text{and} \quad a \cong_\nu b. \tag{9.3}$$

The ***surpassing*** L***-identity*** $f \underset{L}{\models} g$ *holds for* $f, g \in \mathrm{Fun}(R^{(n)}, R)$ *if* $f(a_1, \dots, a_n) \underset{L}{\models} g(a_1, \dots, a_n)$ *for all* $a_1, \dots, a_n \in R$.

The ***surpassing*** (L, ν)***-identity*** $f \underset{L}{\models}_\nu g$ *holds for* $f, g \in \mathrm{Fun}(R^{(n)}, R)$ *if* $f(a_1, \dots, a_n) \underset{L}{\models}_\nu g(a_1, \dots, a_n)$ *for all* $a_1, \dots, a_n \in R$.

9.3.1. *Layered surpassing identities of commutative layered semirings.* Just as the Boolean algebra satisfies the PI $x^2 = x$, we have some surpassing identities for commutative layered domains†.

Proposition 9.13. *(Frobenius identity)* $(x_1 + x_2)^m \underset{L}{\models}_\nu x_1^m + x_2^m$.

Proof. This is just a restatement of [IzKR5, Remark 5.2]. □

Proposition 9.14. $(x_1 + x_2 + x_3)(x_1x_3 + x_2x_3 + x_1x_2) \underset{L}{\models}_\nu (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$. *More generally, let* $g_1 = \sum_i x_i$, $g_2 = \sum_{i<j} x_ix_j$, $\dots$, *and* $g_{m-1} = \sum_i \prod_{j \neq i} x_j$. *Then*

$$g_1 \cdots g_{m-1} \underset{L}{\models}_\nu \prod_{i<j} (x_i + x_j). \tag{9.4}$$

Proof. This is just a restatement of [IzR1, Theorem 8.51]. □

9.4. **Layered surpassing identities of matrices.**

We applied the strong transfer principle of Akian, Gaubert, and Guterman [AkGG, Theorem 4.21] to the (standard) supertropical matrix semiring in [IzR4]. We would like to make a similar argument in the layered case, but must avoid the following kind of counterexamples, pointed out by Adi Niv:

Example 9.15. *Suppose* $A = \begin{pmatrix} {}^{[1]}10 & {}^{[2]}4 \\ {}^{[2]}4 & {}^{[10]}0 \end{pmatrix}$. *Then* $A^2 = \begin{pmatrix} {}^{[1]}20 & {}^{[2]}14 \\ {}^{[2]}14 & {}^{[4]}8 \end{pmatrix}$, *so* $|A| = {}^{[10]}10$ *whereas* $|A^2| = {}^{[8]}28$, *which does not* $\mathbb{N}$*-surpass* $|A|^2$ *(and does not even* $\mathbb{N}$*-surpass* $|A|$*).*

The difficulty in the example was that some ν-small entry of A has a high layer which provides $|A|$ a high layer but does not affect the powers of A. There is a version of surpassing which is useful in this context.

Definition 9.16. *An element $c \in R$ is a **strong ℓ-ghost** (for $\ell \in L_+$) if $s(c) \geq 2\ell$.*

*The **strong ℓ-surpassing relation** $a \underset{S\ell}{\models} b$ holds in an L-layered domain*† *R, if either*

$$\begin{cases} a = b + c & \text{with} \quad c \quad \text{a strong } \ell\text{-ghost} \\ \quad \text{or} & \\ a = b. & \end{cases} \tag{9.5}$$

We often take $\ell = s(b)$. In this case $b + b \underset{S\ell}{\models} b$ (as well as $b + b \underset{\ell}{\models} b$).

*The **strong ℓ-surpassing relation** $(a_{i,j}) \underset{S\ell}{\models} (b_{i,j})$ holds for matrices $(a_{i,j})$ and $(b_{i,j})$, if $a_{i,j} \underset{S\ell}{\models} b_{i,j}$ for each i, j.*

We say that a matrix A is ℓ- **layered** if each entry has layer $\geq \ell$. We are ready for our other two versions of layered identities.

Definition 9.17. *The **strong** (ℓ, d)**-surpassing identity** $f \underset{S\ell;d}{\models} g$ holds for $f, g \in \mathrm{Fun}(M_n(R)^{(m)}, M_n(R))$ if $f(A_1, \dots, A_m) \underset{S\tilde{\ell}}{\models} g(A_1, \dots, A_m)$ with $\tilde{\ell} = \ell^d$, for all ℓ-layered matrices $A_1, \dots, A_m \in M_n(R)$.*

In the standard supertropical theory we take $\ell = \tilde{\ell} = 1$, but in the general layered theory we may need to consider other ℓ. Formally set $P(x_1, \dots, x_\ell) = P^+ - P^-$ and $Q(x_1, \dots, x_\ell) = Q^+ - Q^-$. We say Q is **admissible** if the monomials of Q^+ and Q^- are distinct, for each pair (i, j).

We then obtain the following metatheorem, along the lines of [AkGG] (just as in [IzR4, Theorem 2.4]):

Theorem 9.18. *Suppose $P = Q$ is a homogeneous matrix identity of $M_n(\mathbb{Z})$ of degree d, with Q admissible. Then the matrix semiring*† *$M_n(R)$ satisfies the strong (ℓ, d)-surpassing identity*

$$P^+ + P^- \underset{S\ell;d}{\models} Q^+ + Q^-.$$

Here are some applications.

Corollary 9.19. $|AB| \underset{S\ell;d}{\models} |A|\,|B|$ *for L-layered $n \times n$ matrices A and B, where $d = 2n$.*

Given an L-layered matrix A and the polynomial

$$f_A := |\lambda I + A| = \alpha_n \lambda^n + \cdots + \alpha_1 \lambda + \alpha_0,$$

we define the polynomial $\widetilde{f_A}$ to be

$$\widetilde{f_A} = \widehat{\alpha_n} \lambda^{n-1} + \cdots + \widehat{\alpha_2} \lambda + \widehat{\alpha_1},$$

where $s(\widehat{\alpha_i}) = \ell^{n-i}$ and $\widehat{\alpha_i} \cong_\nu \alpha_i$.

Theorem 9.20. $\widetilde{f_A}(A) \underset{S\ell;d}{\models} \mathrm{adj}(A)$*, where $d = n - 1$, for any ℓ-layered matrix A.*

Proof. This is an identity for $M_n(\mathbb{Z})$, using the usual determinant. □

Proposition 9.21. $\mathrm{adj}(\mathrm{adj}(A)) \underset{S\ell;d}{\models} |A|^{n-2} A$*, where $d = n - 1$, for any ℓ-layered matrix A.*

Questions for further thought:

Q1. What are all the semiring† PIs of $M_n(R)$?

Specifically, we have the Specht-like question:

Q2. Are all semiring† PIs of $M_n(R)$ a consequence of a given finite set?

Example 9.22. *It is shown in* [IzM] *that the semiring of* 2×2 *matrices over the max-plus algebra satisfies the semigroup identity*

$$AB^2A\ AB\ AB^2A = AB^2A\ BA\ AB^2A. \tag{9.6}$$

The way of proving this identity is essentially based on showing that pairs of polynomials corresponding to compatible entries in the right and the left product above define the same function. This identification is performed by using the machinery of Newton polytopes, and thus is valid also for supertropical polynomials. From the results of [IzM]*, we also conclude that this identity is minimal.*

References

[AkBG] M. Akian, R. Bapat, and S. Gaubert. Max-plus algebra, In: Hogben, L., Brualdi, R., Greenbaum, A., Mathias, R. (eds.) *Handbook of Linear Algebra*. Chapman and Hall, London, 2006.

[AkGG] M. Akian, S. Gaubert, and A. Guterman. Linear independence over tropical semirings and beyond. In *Tropical and Idempotent Mathematics*, G.L. Litvinov and S.N. Sergeev, (eds.), *Contemp. Math.* 495:1–38, 2009.

[AmL] S.A¿ Amitsur and J. Levitzki. Minimal identities for algebras. Proc. American Mathematical Society 1, 449–463 1950.

[BiG] R. Bieri and R. Groves, The geometry of the set of characters induced by valuations. *J. für die Reine und angevandte Mathematik*, 374:168–195, 1984.

[Bo] N. Bourbaki, *Alg. Comm. VI*, §3, No.1.

[BrR] R. A. Brualdi and H. J. Ryser. *Combinatorial matrix theory*. Cambridge University Press, 1991.

[CHWW] G. Cortinas, C. Haesemeyer, M. Walker, and C. Weibel, Toric varieties, monoid schemes, and descent. Preprint, 2010.

[DeS] M. Devlin and B. Sturmfels, *Tropical convexity*. Documenta Mathematica 9 (2004), 1–27, Erratum 205–6.

[Gat] A. Gathmann, Tropical algebraic geometry. *Jahresbericht der DMV* 108:3–32, 2006. Preprint at arXiv:math.AG/0601322.

[Gol] J. Golan, *The theory of semirings with applications in mathematics and theoretical computer science*, Vol. 54, Longman Sci & Tech., 1992.

[ItMS] I. Itenberg, G. Mikhalkin and E. Shustin, *Tropical Algebraic Geometry*, Oberwolfach Seminars, 35, Birkhäuser Verlag, Basel, 2007.

[IzKR1] Z. Izhakian, M. Knebusch, and L. Rowen. A Glimpse at Supertropical Valuation theory An. St. Univ. Ovidius Constanta 19, no. 2, 131–142, 2011.

[IzKR2] Z. Izhakian, M. Knebusch, and L. Rowen, Supertropical linear algebra, Preprint at arXiv:1008.0025, 2010.

[IzKR3] Z. Izhakian, M. Knebusch, and L. Rowen, Dual spaces and bilinear forms in supertropical linear algebra, *Linear and Mult. Algebra*, to appear. (Preprint at arXiv:1201.6481.)

[IKR4] Z. Izhakian, M. Knebusch, and L. Rowen, Dominance and transmissions in supertropical valuation theory, *Communications in Algebra*, to appear. (Preprint at arXiv:1102.1520.)

[IzKR4] Z. Izhakian, M. Knebusch, and L. Rowen. Layered tropical mathematics (Submitted May 2012)

[IzKR5] Z. Izhakian, M. Knebusch, and L. Rowen. Categorical layered mathematics Contemporary Mathematics "Proceedings of the CIEM Workshop on Tropical Geometry," to appear.

[IzM] Z. Izhakian and S. W. Margolis. Semigroup identities in the monoid of 2-by-2 tropical matrices. *Semigroup Furom* 80(2), 191–218, 2010.

[IzR1] Z. Izhakian and L. Rowen, Supertropical algebra. *Adv. in Math* 225:4, 2222–2286, 2010.

[IzR2] Z. Izhakian and L. Rowen, The tropical rank of a tropical matrix. *Communications in Algebra*, 37:11, 3912–3927, 2009.

[IzR3] Z. Izhakian and L. Rowen, Supertropical matrix algebra, *Israel J. Math.*, 182(1), 383–424, 2011.

[IzR4] Z. Izhakian and L. Rowen, Supertropical matrix algebra II: Solving tropical equations. *Israel J. Math.* 186(1), 69–97, 2011.

[IzR5] Z. Izhakian and L. Rowen, Supertropical matrix algebra III: Powers of matrices and generalized eigenspaces., *J. of Algebra*, 341(1), 125–149, 2011.

[IzR6] Z. Izhakian and L. Rowen, Supertropical polynomials and resultants, *J. Algebra* 324, 1860–1886, 2010.

[J] Jacobson, N., *Basic Algebra*, Freeman, 1980.

[Ko] V.M. Kopytov, Lattice ordered groups (Russian), Nauka, Moscow (1984)

[L1] G. Litvinov, The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction. *J. of Math. Sciences*, 140(3):426–444, 2007.

[L2] G. Litvinov, Dequantization of mathematical structures and tropical idempotent mathematics: An introductory lecture. (See next reference), 2012

[L3] G. Litvinov and V.P. Maslov, Tropical and idempotent mathematics: International Workshop. Pncelet Laboratory and Moscow Center for Continuous Mathematical Education, Independent University of Moscow, 2012.

[MS] D. Maclagan and B. Sturmfels, *Tropical Geometry*. Preprint, 2009.

[M] D. Marker, Model theory : An introduction Springer Graduate texts in mathematics; 217, 2002.

[Nag] Nagata, to appear,

[Niv] Niv, Adi, Characteric polynomials of supertropical matrices, Commun. in Algebra, to appear, 2012.

[Par] B. Parker. Exploded fibrations, preprint at arXiv: 0705.2408v1, 2007.

[Pay1] S. Payne. Fibers of tropicalizations, Arch. Math., 2010 Correction: preprint at arXiv: ?? [math.AG], 2012.

[Pay2] S. Payne. Analytification is the limit of all tropicalizations, preprint at arXiv: 0806.1916v3 [math.AG], 2009.

[R1] L.H. Rowen. *Graduate Algebra: Commutative View*. Pure and Applied Mathematics 73, Amer. Math. Soc., 2006.

[R2] L.H. Rowen. *Graduate algebra: A noncommutative view Semigroups and Combinatorial Applications*. American Mathematical Society, 2008.

[Sa] G.E. Sacks, Saturated Model Theory, Mathematical Lecture Noets 80 Benjamin, 1972.

[ShSh] E. Sheiner and S. Shnider, An exploded-layered version of Payne's generalization of Kapranov's theorem Preprint, 2012.

[SS] D. E. Speyer and B. Sturmfels, Tropical mathematics, *Math. Mag.*, 82 (2009), 1631/2-173.

[Vi] O. Viro, Hyperfields for tropical geometry I. Hyperfields and dequantization Preprint at arXiv:math.AG/1006.3034v2

[W] C. Weibel, EGA for monoids. Preprint, 2010.

[WW] H.J. Weinert and R. Wiegandt, On the structure of semifields and lattice-ordered groups, *Periodica Mathematica Hungaria 32 (1-2) (1996), 129–147.*