

פתרון תרגיל 3

.1

$$\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1 + \sqrt[3]{x})} dx$$

$$\left\{ \begin{array}{l} x = t^6 \\ dx = 6t^5 dt \end{array} \right\}$$

$$= \int \frac{t^6 + t^4 + t}{t^6(1+t^2)} \cdot 6t^5 dt = 6 \int \frac{t^5 + t^3 + 1}{(1+t^2)} dt = 6 \int t^3 dt + 6 \int \frac{dt}{t^2 + 1}$$

$$= \frac{3}{2} t^4 + 6 \arctan t + c = \frac{3}{2} x^{2/3} + 6 \arctan x^{1/6} + c$$

.2

$$\int \frac{dx}{x + \sqrt{x^2 - x + 1}}$$

נבצע הצבת אוילר לפי $a > 0$

$$t + x = \sqrt{x^2 - x + 1}$$

כלומר

$$t^2 + 2tx + x^2 = x^2 - x + 1$$

$$t^2 + 2tx = -x + 1$$

$$x(2t + 1) = 1 - t^2$$

$$x = \frac{1 - t^2}{2t + 1}$$

$$dx = \frac{-2t(2t + 1) - 2(1 - t^2)}{(2t + 1)^2} dt = \frac{-2t^2 - 2t - 2}{(2t + 1)^2} dt$$

ולכן

$$\begin{aligned} \int \frac{dx}{x + \sqrt{x^2 - x + 1}} &= \int \frac{-2t^2 - 2t - 2}{\frac{1-t^2}{2t+1} + (t + \frac{1-t^2}{2t+1}) (2t + 1)^2} dt \\ &= -2 \int \frac{t^2 + t + 1}{(2t + 1)^2 (t + \frac{2-2t^2}{2t+1})} dt = \\ &= -2 \int \frac{t^2 + t + 1}{4t^3 + 4t^2 + t + (2t + 1)(2 - 2t^2)} dt \\ &= -2 \int \frac{t^2 + t + 1}{4t^3 + 4t^2 + t - 4t^3 - 2t^2 + 4t + 2} dt \\ &= - \int \frac{2t^2 + 2t + 2}{2t^2 + 5t + 2} dt \end{aligned}$$

נבצע חילוק פולינומים

$$-\frac{2t^2 + 2t + 2}{2t^2 + 5t + 2} = -1 + \frac{3t}{2t^2 + 5t + 2}$$

כמוכן ש

$$\int -1 = -t + C$$

נותר להסתכל על

$$\int \frac{3t}{2t^2 + 5t + 2} dt$$

נפרק לשברים חלקיים

$$\frac{3t}{2t^2 + 5t + 2} = \frac{\frac{3}{2}t}{(t + \frac{1}{2})(t + 2)} = \frac{A}{t + \frac{1}{2}} + \frac{B}{t + 2} = \frac{(A + B)t + 2A + \frac{1}{2}B}{(t + \frac{1}{2})(t + 2)}$$

מקבלים משוואות

$$2A + \frac{1}{2}B = 0$$

$$A + B = \frac{3}{2}$$

שפתרון:

$$A = -\frac{1}{2} \quad B = 2$$

ולכן

$$\begin{aligned} \int \frac{3t}{2t^2 + 5t + 2} dt &= \int \frac{-\frac{1}{2}}{t + \frac{1}{2}} dt + \int \frac{2}{t + 2} dt \\ &= -\frac{1}{2} \ln |t + \frac{1}{2}| + 2 \ln |t + 2| + C \end{aligned}$$

נאחד הכל חזרה ונציב $t = \sqrt{x^2 - x + 1} - x$ כדי לקבל

$$\begin{aligned} \int \frac{dx}{x + \sqrt{x^2 - x + 1}} &= -t - \frac{1}{2} \ln |t + \frac{1}{2}| + 2 \ln |t + 2| + C = \\ &= x - \sqrt{x^2 - x + 1} - \frac{1}{2} \ln |\sqrt{x^2 - x + 1} - x + \frac{1}{2}| + 2 \ln |\sqrt{x^2 - x + 1} - x + 2| + C \end{aligned}$$

.3

$$\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$$

נבצע הצבת אוילר

$$\sqrt{1+x-x^2} = tx + 1$$

$$1+x-x^2 = t^2x^2 + 2tx + 1$$

$$x-x^2 = t^2x^2 + 2tx$$

$$1 - x = t^2x + 2t$$

$$x(t^2 + 1) = 1 - 2t$$

$$x = \frac{1 - 2t}{t^2 + 1}$$

$$dx = \frac{-2(t^2 + 1) - (1 - 2t) \cdot 2t}{(t^2 + 1)^2} dt = \frac{2t^2 - 2t - 2}{(t^2 + 1)^2} dt$$

$$1 + x = 1 + \frac{1 - 2t}{t^2 + 1} = \frac{t^2 + 1 + 1 - 2t}{t^2 + 1} = \frac{t^2 - 2t + 2}{t^2 + 1}$$

נציב באינטגרל ונקבל

$$\begin{aligned} \int \frac{dx}{(1+x)\sqrt{1+x-x^2}} &= \int \frac{(2t^2 - 2t - 2)(t^2 + 1)}{(t^2 + 1)^2(t^2 - 2t + 2)\left(\frac{t(1-2t)}{t^2+1} + 1\right)} dt \\ &= \int \frac{(2t^2 - 2t - 2)}{(t^2 - 2t + 2)(t - 2t^2 + t^2 + 1)} dt \\ &= 2 \int \frac{(t^2 - t - 1)}{(t^2 - 2t + 2)(t^2 - t - 1)} dt \\ &= -2 \int \frac{dt}{t^2 - 2t + 2} \\ &= -2 \int \frac{dt}{(t-1)^2 + 1} = -2 \arctan(t-1) + C \\ &= -2 \arctan\left(\frac{\sqrt{1+x-x^2}-1}{x} - 1\right) + C \end{aligned}$$

$$\int \frac{xdx}{\sqrt{(7x-10-x^2)^3}}$$

$$-x^2 + 7x - 10 = 0$$

$$x^2 - 7x + 10 = 0$$

$$x_1 = 2 \quad x_2 = 5$$

$$\sqrt{7x-10-x^2} = t(x-2)$$

$$\sqrt{-(x-2)(x-5)} = t(x-2)$$

$$-(x-2)(x-5) = t^2(x-2)^2$$

$$-(x-5) = t^2(x-2)$$

$$x(t^2+1) = 5 + 2t^2$$

$$x = \frac{5 + 2t^2}{t^2 + 1}$$

$$dx = \frac{4t(t^2+1) - (5+2t^2) \cdot 2t}{(t^2+1)^2} dt = \frac{-6t}{(t^2+1)^2} dt$$

$$\begin{aligned}
&\Rightarrow \int \frac{5+2t^2}{(t^2+1)t^3 \left[\frac{5+2t^2}{t^2+1} - 2 \right]^3} \cdot \frac{(-6t)}{(t^2+1)^2} dt \\
&= -6 \int \frac{(5+2t^2) dt}{t^2(5+2t^2-2t^2-2)^3} dt = -\frac{2}{9} \int \frac{5+2t^2}{t^2} dt = \frac{10}{9} \frac{1}{t} - \frac{4}{9} t + c \\
&= \frac{10}{9} \cdot \frac{(x-2)}{\sqrt{7x-10-x^2}} - \frac{4}{9} \frac{\sqrt{7x-10-x^2}}{x-2} + c
\end{aligned}$$

.5

$$\begin{aligned}
&\int \frac{2}{(2-x)^2} \sqrt[3]{\frac{2-x}{2+x}} dx \\
&\frac{2-x}{2+x} = t^3 \\
&2-x = (2+x)t^3 \\
&2-2t^3 = x(t^3+1) \\
&x = \frac{2-2t^3}{t^3+1} = 2 \cdot \frac{1-t^3}{t^3+1} \\
&dx = 2 \cdot \frac{(-3t^2(t^3+1) - 3t^2(1-t^3))}{(t^3+1)^2} dt = \frac{-12t^2}{(t^3+1)^2} dt \\
&2-x = 2 - \frac{2(1-t^3)}{(t^3+1)} = \frac{4t^3}{t^3+1} \\
&= \int \frac{2(t^3+1)^2}{16t^6} \cdot t \cdot \left(\frac{-12t^2}{(t^3+1)^2} \right) dt \\
&= \int \frac{-24t^3}{16t^6} dt = -\frac{3}{2} \int \frac{dt}{t^3} = \frac{3}{4} \frac{1}{t^2} + c = \frac{3}{4} \left(\frac{2+x}{2-x} \right)^{\frac{2}{3}} + c
\end{aligned}$$

.6

$$\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx = \frac{1}{4} \int \frac{\cos 2x}{\sin^2 2x} dx$$

נציב $t = \sin 2x$ ואז

$$dt = 2 \cos 2x dx$$

כלומר

$$\frac{1}{4} \int \frac{\cos 2x}{\sin^2 2x} dx = \frac{1}{8} \int \frac{dt}{t^2} = -\frac{1}{8t} + C = -\frac{1}{8 \sin 2x} + C$$

.7

$$\int \frac{1}{1 + \sin x + \cos x} dx$$

נשתמש בהצבה אוניברסלית ונקבל

$$\begin{aligned} \int \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt &= \int \frac{1+t^2}{1+t^2+2t+1-t^2} \frac{2}{1+t^2} dt \\ &= \int \frac{2}{2+2t} dt = \ln(1+t) + C = \ln\left(\tan \frac{x}{2} + 1\right) + C \end{aligned}$$

.8

$$\int \frac{\sin 2x}{1 + \sin^2 x} dx = \int \frac{2 \sin x \cos x}{1 + \sin^2 x} dx$$

נציב $t = \sin x$ ונקבל

$$\int \frac{2t}{1+t^2} dt = \ln(1+t^2) = \ln(1 + \sin^2 x)$$