8. Delilah wore a ring on every finger, and had a finger in every pie. (*d*-Delilah, *Rx*-*x* is a ring, *Fxy*-*x* is a finger of *y*, *Oxy*-*x* is on *y*, *Px*-*x* is a pie, *Ixy*-*x* is in *y*.)
9. The race is not always to the swift, nor the battle to the strong. (*Rx*-*x* is a race, *Sx*-*x* is swift, *Bx*-*x* is a battle, *Kx*-*x* is strong, *Wxy*-*x* wins *y*.)
10. Anyone who accomplishes anything will be envied by everyone. (*Px*-*x* is a person, *Axy*-*x* accomplishes *y*, *Exy*-*x* envies *y*.)
11. To catch a fish one must have some bait. (*Px*-*x* is a person, *Fx*-*x* is a fish, *Bx*-*x* is bait, *Cxy*-*x* catches *y*, *Hxy*-*x* has *y*.)
12. Every student does some problems, but no student does all of them. (*Sx*-*x* is a student, *Px*-*x* is a problem, *Dxy*-*x* does *y*.)
13. Any contestant who answers all the questions put to him will win any prize he chooses. (*Cx*-*x* is a contestant, *Qx*-*x* is a question, *Px*-*x* is a prize, *Axy*-*x* answers *y*, *Pxy*-*x* is put to *y*, *Wxy*-*x* wins *y*, *Cxy*-*x* chooses *y*.)
14. Every son has a father but not every father has a son. (*Px*-*x* is a person, *Mx*-*x* is male, *Pxy*-*x* is a parent of *y*.)
15. A person is maintaining a nuisance if he has a dog who barks at every stranger. (*Px*-*x* is a person, *Nx*-*x* is a nuisance, *Mxy*-*x* maintains *y*, *Dx*-*x* is a dog, *Bxy*-*x* barks at *y*, *Kxy*-*x* knows *y*, *Hxy*-*x* has *y*.)
16. A doctor has no scruples who treats a patient who has no ailment. (*Dx*-*x* is a doctor, *Sx*-*x* is a scruple, *Hxy*-*x* has *y*, *Px*-*x* is a patient, *Ax*-*x* is an ailment, *Txy*-*x* treats *y*.)
17. A doctor who treats a person who has every ailment has a job no one would envy him. (*Dx*-*x* is a doctor, *Px*-*x* is a person, *Txy*-*x* treats *y*, *Ax*-*x* is an ailment, *Hxy*-*x* has *y*, *Jx*-*x* is a job, *Exyz*-*x* envies *y* his *z*.)
18. If a farmer keeps only hens, none of them will lay eggs that are worth setting. (*Fx*-*x* is a farmer, *Kxy*-*x* keeps *y*, *Hx*-*x* is a hen, *Ex*-*x* is an egg, *Lxy*-*x* lays *y*, *Wx*-*x* is worth setting.)

In symbolizing the following, use only the abbreviations: *Px*-*x* is a person, *Sx*-*x* is a store, *Bxyz*-*x* buys *y* from *z*.

19. Everyone buys something from some store (or other).
20. There is a store from which everyone buys something (or other).
21. Some people make all their purchases from a single store.
22. No one buys everything that it sells from any store.
23. No one buys things from every store.



24. No store has everyone for a customer.
25. No store makes all its sales to a single customer.

## II. ARGUMENTS INVOLVING RELATIONS

No new principles need be introduced to deal with relational arguments. The original list of valid argument forms, together with the strengthened method of Conditional Proof and our four quantification rules, enable us (if we have sufficient ingenuity) to construct a demonstration of the validity of every valid argument in which only individual variables are quantified and only truth-functional connectives occur.

However, a certain change of technique is advisable in working with arguments involving relations. In all our previous sample demonstrations, **UI** and **EI** were used to instantiate with respect to a variable different from any quantified in the premiss, and **UG** and **EG** were used to quantify with respect to a variable different from any which occurred free in the premiss. Our inferences were of the following forms:

$$\frac{(x)Fx}{\therefore\ Fy},\quad \frac{(\exists x)Fx}{\therefore\ Fz},\quad \frac{Fx}{\therefore\ (y)Fy},\quad \frac{Fy}{\therefore\ (\exists w)Fw}.$$

But our statement of the quantification rules does not require that $\mu$ and $\nu$ be different variables; they may well be the same. And on the whole it is simpler (wherever it is legitimate) to instantiate with respect to the same variable that had been quantified, and to quantify with respect to the same variable that had been free in the premiss. Thus the above inferences may also take the following forms:

$$\frac{(x)Fx}{\therefore\ Fx},\quad \frac{(\exists x)Fx}{\therefore\ Fx},\quad \frac{Fx}{\therefore\ (x)Fx},\quad \frac{Fy}{\therefore\ (\exists y)Fy}.$$

In this way instantiation is accomplished by simply dropping a quantifier, and generalization is accomplished by simply adding a quantifier. Of course our restrictions on the quantification rules must still be observed. For example, where we have two premisses '$(\exists x)Fx$' and '$(\exists x) \sim Fx$', we can instantiate with respect to one by simply dropping the quantifier, but when this is done, if **EI** is subsequently used on the other, a new variable must be used instead of '$x$', for the latter will already have a free occurrence in the proof under construction. Of course we remain perfectly free to use **UI** to instantiate with respect to any particular variable or constant we choose. The preceding remarks can be illustrated by constructing a demonstration of validity for the argument

There is a man whom all men despise.
Therefore at least one man despises himself.

Its symbolic translation and proof, using '$Mx$' and '$Dxy$' to abbreviate '$x$ is a man' and '$x$ despises $y$' may be written as follows:

| | |
|---|---|
| 1. $(\exists x)[Mx \cdot (y)(My \supset Dyx)] / \therefore (\exists x)(Mx \cdot Dxx)$ | |
| 2. $Mx \cdot (y)(My \supset Dyx)$ | 1, **EI** |
| 3. $(y)(My \supset Dyx)$ | 2, Simp. |
| 4. $Mx \supset Dxx$ | 3, **UI** |
| 5. $Mx$ | 2, Simp. |
| 6. $Dxx$ | 4, 5, M.P. |
| 7. $Mx \cdot Dxx$ | 5, 6, Conj. |
| 8. $(\exists x)(Mx \cdot Dxx)$ | 7, **EG** |

In the foregoing proof, the only use of a quantification rule which was accompanied by a change of variable was the use of **UI** in going from step 3 to step 4, which was done because we needed the expression '$Dxx$' thus obtained.

Another sample demonstration will be given, this time to establish the validity of the third specimen argument stated at the beginning of the present chapter. Its premiss, 'All horses are animals' will be symbolized as '$(x)(Ex \supset Ax)$', where '$Ex$' and '$Ax$' abbreviate '$x$ is a horse' and '$x$ is an animal', respectively. In its conclusion

The head of a horse is the head of an animal

the word 'the' has the same sense that it does in such propositions as 'The whale is a mammal' or 'The burnt child dreads the fire'.



We may paraphrase it therefore first as

All heads of horses are heads of animals.

then as

$(x)[(x \text{ is the head of a horse}) \supset (x \text{ is the head of an animal})]$.

and finally, writing '$Hxy$' for '$x$ is the head of $y$', we may express the conclusion by the formula

$$(x)[(\exists y)(Ey \cdot Hxy) \supset (\exists y)(Ay \cdot Hxy)].$$

Once it is symbolized, the argument is easily proved valid by the techniques already available:

| | |
|---|---|
| 1. $(x)(Ex \supset Ax) / \therefore (x)[(\exists y)(Ey \cdot Hxy) \supset (\exists y)(Ay \cdot Hxy)]$ | |
| →2. $(y) \sim(Ay \cdot Hxy)$ | |
| 3. $\sim(Ay \cdot Hxy)$ | 2, **UI** |
| 4. $\sim Ay \vee \sim Hxy$ | 3, De M. |
| 5. $Ay \supset \sim Hxy$ | 4, Impl. |
| 6. $Ey \supset Ay$ | 1, **UI** |
| 7. $Ey \supset \sim Hxy$ | 6, 5, H.S. |
| 8. $\sim Ey \vee \sim Hxy$ | 7, Impl. |
| 9. $\sim(Ey \cdot Hxy)$ | 8, De M. |
| 10. $(y) \sim(Ey \cdot Hxy)$ | 9, **UG** |
| 11. $(y) \sim(Ay \cdot Hxy) \supset (y) \sim(Ey \cdot Hxy)$ | 2–10, C.P. |
| 12. $\sim(y) \sim(Ey \cdot Hxy) \supset \sim(y) \sim(Ay \cdot Hxy)$ | 11, Trans. |
| 13. $(\exists y)(Ey \cdot Hxy) \supset (\exists y)(Ay \cdot Hxy)$ | 12, **QN** |
| 14. $(x)[(\exists y)(Ey \cdot Hxy) \supset (\exists y)(Ay \cdot Hxy)]$ | 13, **UG** |

Again, the only time a change of variables accompanied the use of a quantification rule (in step 6) was when the change of variable was needed for subsequent inferences.

The first specimen argument presented in this chapter, which dealt with the relation of *being older than*, raises a new problem, which will be discussed in the following section.

## EXERCISES

Construct a formal proof of validity for each of the following arguments:

1. Whoever supports Ickes will vote for Jones. Anderson will vote for no one but a friend of Harris. No friend of Kelly has Jones for a friend. Therefore, if Harris is a friend of Kelly, Anderson will not support Ickes. (*Sxy*-*x* supports *y*, *Vxy*-*x* votes for *y*, *Fxy*-*x* is a friend of *y*, *a*-Anderson, *i*-Ickes, *j*-Jones, *h*-Harris, *k*-Kelly.)
2. Whoever belongs to the Country Club is wealthier than any member of the Elks Lodge. Not all who belong to the Country Club are wealthier than all who do not belong. Therefore not everyone belongs either to the Country Club or the Elks Lodge. (*Cx*-*x* belongs to the Country Club, *Ex*-*x* belongs to the Elks Lodge, *Px*-*x* is a person, *Wxy*-*x* is wealthier than *y*.)
3. All circles are figures. Therefore all who draw circles draw figures. (*Cx*-*x* is a circle, *Fx*-*x* is a figure, *Dxy*-*x* draws *y*.)
4. There is a professor who is liked by every student who likes any professor at all. Every student likes some professor or other. Therefore there is a professor who is liked by all students. (*Px*-*x* is a professor, *Sx*-*x* is a student, *Lxy*-*x* likes *y*.)
5. Only a fool would lie about one of Bill's fraternity brothers to him. A classmate of Bill's lied about Al to him. Therefore if none of Bill's classmates are fools, then Al is not a fraternity brother of Bill. (*Fx*-*x* is a fool, *Lxyz*-*x* lies about *y* to *z*, *Cxy*-*x* is a classmate of *y*, *Bxy*-*x* is a fraternity brother of *y*, *a*-Al, *b*-Bill.)
6. It is a crime to sell an unregistered gun to anyone. All the weapons that Red owns were purchased by him from either Lefty or Moe. So if one of Red's weapons is an unregistered gun, then if Red never bought anything from Moe, Lefty is a criminal. (*Rx*-*x* is registered, *Gx*-*x* is a gun, *Cx*-*x* is a criminal, *Wx*-*x* is a weapon, *Oxy*-*x* owns *y*, *Sxyz*-*x* sells *y* to *z*, *r*-Red, *l*-Lefty, *m*-Moe.)
7. No one respects a person who does not respect himself. No one will hire a person he does not respect. Therefore a person who respects no one will never be hired by anybody. (*Px*-*x* is a person, *Rxy*-*x* respects *y*, *Hxy*-*x* hires *y*.)
8. Everything on my desk is a masterpiece. Anyone who writes a masterpiece is a genius. Someone very obscure wrote some of the novels on my desk. Therefore some very obscure person is a genius. (*Dx*-*x* is on my desk, *Mx*-*x* is a masterpiece, *Px*-*x* is a person, *Gx*-*x* is a genius, *Ox*-*x* is very obscure, *Nx*-*x* is a novel, *Wxy*-*x* wrote *y*.)
9. Any book which is approved by all critics is read by every literary person. Anyone who reads anything will talk about it. A critic will



approve any book written by any person who flatters him. Therefore if someone flatters every critic then any book he writes will be talked about by all literary persons. (*Bx*-*x* is a book, *Cx*-*x* is a critic, *Lx*-*x* is literary, *Px*-*x* is a person, *Axy*-*x* approves *y*, *Rxy*-*x* reads *y*, *Txy*-*x* talks about *y*, *Fxy*-*x* flatters *y*, *Wxy*-*x* writes *y*.)

10. A work of art which tells a story can be understood by everyone. Some religious works of art have been created by great artists. Every religious work of art tells an inspirational story. Therefore if some people admire only what they cannot understand, then some artists' creations will not be admired by everyone. (*Ax*-*x* is an artist, *Gx*-*x* is great, *Px*-*x* is a person, *Sx*-*x* is a story, *Ix*-*x* is inspirational, *Rx*-*x* is religious, *Wx*-*x* is a work of art, *Cxy*-*x* creates *y*, *Axy*-*x* admires *y*, *Txy*-*x* tells *y*, *Uxy*-*x* can understand *y*.)

## III. SOME PROPERTIES OF RELATIONS

There are a number of interesting properties that relations themselves may possess. We shall consider only a few of the more familiar ones, and our discussion will be confined to properties of *dyadic* relations.

Dyadic relations may be characterized as *symmetrical*, *asymmetrical*, or *non-symmetrical*. Various symmetrical relations are designated by the phrases: 'is next to', 'is married to', and 'has the same weight as'. A *symmetrical* relation is one such that if one individual has that relation to a second individual, then the second individual must have that relation to the first. A propositional function '*Rxy*' designates a symmetrical relation if and only if

$$(x)(y)(Rxy \supset Ryx).$$

On the other hand, an *asymmetrical* relation is one such that if one individual has that relation to a second individual, then the second individual *cannot* have that relation to the first. Various asymmetrical relations are designated by the phrases: 'is north of', 'is parent of', and 'weighs more than'. A propositional function '*Rxy*' designates an asymmetrical relation if and only if

$$(x)(y)(Rxy \supset \sim Ryx).$$