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אוניברסיטת בר-אילן
המחלקה למתמטיקה

Structural foundation of supertropical algebra

Research Proposal

יסודות המבנה של האלגברה הסופרטרופית

הצעת מחקר

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1 Introduction

Tropical algebraic geometry is a field of mathematics that has been growing during the last 10 years. The main purpose is to generalize the process of translating complicated geometric problems into solvable combinatorial problems. This idea is achieved by matching regular geometric object with piece-wise linear ones.

Let us define the main algebraic structure of tropical geometry, the max-plus algebra:

$$a \oplus b = max(a, b) \tag{1}$$

$$a \odot b = a + b \tag{2}$$

We wish to study algebraic geometry over the max-plus algebra. In order to do so, we should look at roots of polynomials. However, the usual definition of roots is useless since zero does not play its classical role in this algebra. For instance, let us look at the max-plus polynomial $p(\lambda) = \lambda^2 + 4\lambda + 5$. The equation $p(\lambda) = -\infty$ is false for every λ.

Looking at the graph of the polynomial p, we can see that it is comprised of three line segments. The line segments meet at the points 1 and 4; thus we would expect these points to be the roots of p. One can also notice that p can be factored as $p = (\lambda + 1)(\lambda + 4)$. These facts gave rise to the idea of supertropical geometry by Zur Izhakian in [5].

We investigate these possible roots and see that they come about when two monomials are equal since these are the points where the graph of the polynomial changes its slope. Therefore Izhakian has built a structure that sends the sum of two equal elements to a "ghost" element. We treat these ghosts as zeros since we wish the said points to be roots. This structure is the foundation of the supertropical geometry and will be defined later in the proposal.

The notion of supertropical algebra is a way to formalize some of the results of tropical geometry, to prove classical theorems that could not be even formulated in tropical geometry and hopefully solve open classical problems. This field of mathematics is new and exciting as an area that has never been explored.

2 Tropical Geometry

Tropical geometry was described in the introduction as a general theory that enables one to translate complicated geometric problems into a solvable combinatorial one. As a derivative of this broad definition there is more than one way to approach tropical geometry. Here we describe one approach [1].

Amoebas are the geometrical objects defined as images of varieties by the function $F : \mathbb{C}^n \to \mathbb{R}^n$ given by

$$F(z_1, z_2, ..., z_n) = (log|z_1|, log|z_2|, ..., log|z_n|)$$

These amoebas have different asymptotic limits in different directions, and the idea is to take the lines to which the amoeba converges. We achieve this goal by narrowing the amoeba to have zero width.

Tropical curves are defined as the limit obtained when sending F to $-\log_t$ instead of log, as t tends to zero. These tropical curves are piecewise linear objects as we wished.

We see that the max-plus behavior derives from the definition of the tropical curves as limits of the log action on regular curves. The log of a sum of two numbers tends to the log of the maximum of the summands, and the log of a product of two numbers is the sum of the logs of these numbers.

3 SuperTropical Geometry

We will now formalize the notions presented in the introduction, based on the paper of Izhakian and Rowen [2].

3.1 Extended Tropical Arithmetic

In his paper [5], Izhakian has introduced the following structure $\mathbb{T}$ which is the disjoint union of $\mathbb{R}$ (called tangibles) and its copy denoted $\mathbb{R}^\nu$ (called ghosts) and the formal element $-\infty$. We define the following valuation $v(x) = x^\nu$. We endow $\mathbb{T}$ with the two operations $\oplus$ and $\odot$:

$$x \oplus y = \begin{cases} x & \text{if} \quad v(x) > v(y) \\ y & \text{if} \quad v(x) < v(y) \\ v(x) & \text{if} \quad v(x) = v(y) \end{cases} ,$$

$x \odot y = x + y$ for all $x, y \in \mathbb{R}$,

$x \odot g = g \odot x = v(x) + g$ for all $x \in \mathbb{R}, g \in \mathbb{R}^\nu$.

One can see that the ghost part together with the infinity element form an ideal of $\mathbb{T}$. These ghosts elements were added as means to study the max-plus algebra more thoroughly. However, the structure of the ghost part is very intriguing and its essence is still not completely clear. First we look at the ghost part as an extension of zero in the sense that each element with a ghost image is considered a root. We will discuss later in this proposal the notion of encapsulating multiplicity of a root into the ghost part as well.

3.2 Polynomials

We wish to study the theory of polynomials in order to study varieties as roots of polynomials. Immediately we notice that unlike regular polynomials, different polynomials may be equal as functions or *essential-equal*. For example: $\lambda^2 + 4\lambda + 10 = \lambda^2 + 5\lambda + 10 = \lambda^2 + 10$. In order to get classical tools such as unique factorization and a derivative, we must choose the right member of each functional class of polynomials (which we define later). The full polynomial [2] is a very good candidate for such a representing member.

Formalizing the notion of roots from the introduction, we define the roots of a polynomial f to be the points a so that $f(a)$ is a ghost.

We define the essential monomial h of a polynomial f to be a monomial which contribute to the polynomial in the sense that if $f = h + g$ then $h(a)^\nu > g(a)^\nu$ for some a. If one removes an essential polynomial, then by definition the new polynomial will not be equal as a function to the original one. A polynomial is called essential-tangible if the coefficient of its essential monomials are all tangible. Even though the ghost structure is worth studying in itself, when studying tropical geometry we are more interested in essential-tangible polynomials. Every full polynomial in one indeterminate λ which is essential-tangible is known to be factorized into the product of linear terms $\lambda + r$ where r runs over the roots of the polynomial (including multiplicity). This phenomenon is quite amazing given the odd definition of roots as members with ghost images.

Another reason for choosing the full polynomial as a representative of the polynomial coset will be shown in the following text, in order for our notion of derivative to satisfy one of the basic rules of calculus.

3.3 Other Fundamental Definitions

3.3.1 Topology

The topology of the supertropical structure is built from two types of open sets:

$$\{x \in \mathbb{T} | a^\nu < x^\nu < b^\nu\}$$

$$\{x \in \mathbb{R} | a < x < b\}$$

for all a, b.

3.3.2 Parity

Two members are said to have the same **parity** if both are ghost, or both are tangible.

3.3.3 Functional Classes

We mentioned that different polynomials may be equal as functions; we therefore define functional classes of polynomials. Polynomials belong to the same class iff they are equal as functions.

4 Research Directions

4.1 Integration and Measure Theory

Many of the important results of tropical geometry are related to the Jacobian and integration. We wish to formalize the notation of integration so it will coincide with the known results [6], and then be used as a practical tool to get new results.

4.2 Differentiation

Similar to integration, important work has been done relating to differentiation in tropical geometry. Mikhalkin built his theory using differential geometry and we would like to complement this theory with an analytic derivative. We believe this is an important part of building a full theory that will enable us to formalize and calculate problems more easily in the supertropical algebra.

The classic derivative is defined as the limit of the distance between two values of a function at two points divided by the length between those two points as they get closer. In the supertropical or the tropical structure we have no such distance function. Indeed the natural metric of an element and its ghost would be zero even though they are not the same member.

When studying supertropical matrices, Rowen and Izhakian ([4]) used an axiomatic approach to the study of determinants. One may use that same method in order to study the differential theory of supertropical functions (especially polynomials), using the axioms of derivation. Consider the following axiom: $(fg)' = f'g + g'f$. Applying it to λ^2 we get $(\lambda^2)' = (\lambda\lambda)' = \lambda'\lambda + \lambda\lambda' = (\lambda'\lambda)^{\nu}$. Using induction we can see that the derivative of any non-linear monomial must be a ghost. This leads us to a modification of the supertropical structure.

4.3 Expanded (or Graded) Supertropical Algebra

In the supertropical algebra the sum of two equal tangibles is a ghost, which is one way that roots of polynomials occur. For example, 5 is the root of $\lambda + 5$ since $5 + 5 = 5^{\nu}$; what about $5 + 5 + 5$? We would like that the multiplicity of the root will be reflected by the "level" of its ghost. Consider the polynomial $\lambda^2 + 2^{\nu}\lambda + 4$ with the root $\lambda = 2$. We see that $f(\lambda) = 2 + 2^{\nu} + 2 = 2^{\nu} + 2^{\nu}$ which we may consider as a ghost of a new level. Indeed, the multiplicity of 2 is 2 as $f(\lambda) = (\lambda + 2)^2$. We expand this notion in the section on preliminary results.

Another algebraic consequence that we hope to achieve in this expanded structure is unique factorization of polynomials. In their paper [2], Izhakian and Rowen have shown that unique factorization fails completely in more then one indeterminate:

$$(f_1 + f_2 + f_3)(f_1 f_2 + f_1 f_3 + f_3 f_2) = (f_1 + f_2)(f_1 + f_3)(f_3 + f_2)$$

However, in the expanded structure there is a difference between these two polynomials; The ghost level of the point $(0, 0, 0)$ is 9 at the first polynomial and 8 at the later. Unique factorization will be an important result, and therefore it is one of the first goals of the research.

A third, more geometrical, reason for refining the level of the ghost structure is to be able to distinguish between intersections of three lines or more. The graph of the supertropical polynomial is a polytope in which each face matches a monomial of the polynomial. When two faces meet we get a face of a lower dimension with a ghost value. By defining different levels of ghosts, we can tell the number of faces that created the given face and therefore its dimension. This could provide a new invariant for varieties.

One should note that the equivalence classes of polynomials are subdivided further in this expanded algebra. Thus the equivalence classes may at times be more useful as classes of functions in the ungraded supertropical algebra algebra rather than in the expanded algebra.

5 Preliminary Results

5.1 Essential Equality

As mentioned above, since different polynomials may take precisely the same values in our algebra, we look at the polynomials as functions. Therefore, instead of looking at individual polynomials we look at classes of polynomials, all of which take the same values. Remembering the origin of the supertropical structure, we wonder if two functions can take the same values on the tangibles but different values on the ghosts (and therefore belong to two different classes).

Lemma 5.1. *Let f and g be two continuous functions such that $f(a) = g(a)$ for all a tangible. Then $f^\nu(a) = g^\nu(a)$ for all a.*

Proof. First we see that the tangibles are dense as a set in our topology since each sequence with a tangible limit t also has the ghost t^ν as a limit. Indeed, each open set containing a ghost g also contains all the tangibles with the same ghost value as g. Clearly two members a and b cannot both be the limit of the same sequence if $a^\nu \neq b^\nu$.

Now, since f is continuous its values on the ghosts can be obtained as the limits of its values on certain tangibles (since the tangibles are dense). Thus by fixing f on the tangibles we can determine the rest of f up to parity. □

We can also learn from the proof that by changing the parity of one point of a continuous function from ghost to tangible we do not hurt continuity. Therefore there are functions that take the same values on the tangibles and belong to different classes. However we do have the next positive result:

Theorem 5.2. *Let f and g be two polynomials that are equal on all of the tangibles. Then $f = g$ everywhere.*

Proof. The conditions of the lemma above apply here, and therefore f and g have the same values, differing at most by parity. Moreover, we know that f and g have precisely the same values over the tangibles. Thus, we only have to prove that the parity of g and f is equal over the ghosts.

Assume that f and g do not have a tangible constant term. In that case f and g sends ghosts to ghosts and therefore are equal everywhere.

Now assume that f and g have different constant terms. If we take a to be a small enough tangible, then $f(a) \neq g(a)$, which is absurd.

We are left with f and g both having the same tangible constant term c; whenever f and g have a ν-value bigger than c, then it must be a ghost. We are left with ghosts a^ν where $f(a^\nu) = c^\nu$, in which case at least one monomial of f must equal c in ν-value, that monomial must get the same ν-value for a (and the same is true for any monomial). Therefore $f(a) = c^\nu$ and so $g(a) = f(a) = c^\nu$ and thus $g(a^\nu) = c^\nu$. The same argument applies for g so we get f is tangible iff g is tangible. Together with the lemma $f = g$ as needed. □

5.2 Expanded Supertropical Algebra

As we have seen, we would want that the sum of two ghosts of the same level create a ghost of a higher multiplicity. Let us focus on the case $(\lambda + a)^n$ which is a polynomial with root a of multiplicity n. We shall calculate the level of the ghost of the root a: $(\lambda + a)^n = \sum \binom{n}{i} \lambda^{n-i} a^i$. (Note that the binomial coefficients mean 'times' rather than multiple, since multiple in the tropical sense is addition in the classical sense). When $\lambda = a$ the expression equals $\sum \binom{n}{i} a^n$, which equals to the sum of a^n 2^n times. This means that a is a root with ghost level 2^n. We therefore define the multiplicity of the root as a $\log_2$ of the ghost level of its image. This makes sense since the sum of two ghosts of the same multiplicity creates a ghost of a new multiplicity, and therefore the level is exponential.

We saw that the derivative of a monomial λ^n is inductively shown to be of ghost level n. This means that almost every derivative is a ghost, thus we have a problem to define integration for a tangible polynomial. A somewhat natural idea is to add a negative level of ghosts which will be the coefficients of an integrated tangible polynomial. However, since we have seen that the ghosts level are an exponent of the multiplicity, negative levels are impossible.

Now we would like to define those "negative" ghost levels. Ghost level 1 is defined to be a tangible, tangibles are from multiplicity 0. What is the ghost level of an element of multiplicity -1? It is $2^{-1} = \frac{1}{2}$. We therefore get the further ghost levels $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$... The multiplicity is still the $\log_2$ of the ghost level.

5.3 Differentiation

Izhakian and Rowen have defined the derivative of a monomial $\alpha\lambda^k$ to be $\alpha^{(k)}\lambda^{k-1}$, where $\alpha^{(k)}$ is a ghost of level k. We will use the Frobenius action obtained in [2] in order to see if the following fundamental result from calculus is true in supertropical settings. Let f and g be supertropical polynomials. We wish to know if $[f(g(\lambda))]' = g'(\lambda) \cdot f'(g(\lambda))$.

Since it is known that $(ah_1 + bh_2)' = ah_1' + bh_2'$ for all polynomials h_1, h_2, we may safely assume that $f = \lambda^k$ for some natural k. Now we use the Frobenius action to see that

$$f(g(\lambda)) = (g(\lambda))^k = (\sum a_i\lambda^i)^k = \sum (a_i\lambda^i)^k = \sum a_i^k \lambda^{ik}$$

Therefore $[f(g(\lambda))]' = \sum a_i^{k,(ik)}\lambda^{ik-1}$. (Here $a^{m,(n)}$ means a^m of ghost level n). Now $g' = \sum a_i^{(i)}\lambda^{i-1}$ and $f' = 0^{(k)}\lambda^{k-1}$.

$$f'(g) = 0^{(k)}g^{k-1} = \sum a_i^{k-1,(k)}\lambda^{ik-i}$$

Next we check if the following equation holds

$$g' \cdot f'(g) = \left[\sum a_i^{(i)}\lambda^{i-1}\right] \cdot \left[\sum a_i^{k-1,(k)}\lambda^{ik-i}\right]$$

When we take the sum of the diagonal we get $\sum a_i^{k,(ik)}\lambda^{ik-1} = [f(g)]'$ as we desired. However, we obtain extra monomials which are not necessarily inessential.

This disappointing result is due to a simple fact:

Lemma 5.3. *The derivatives of two essential-equal polynomials are not necessarily essential-equal*

Proof. We will look at the simplest example: $(\lambda + \alpha)^2$. Then $(\lambda^2 + \alpha^2)' = 0^{(2)}\lambda$ while $(\lambda^2 + \alpha^{(2)}\lambda + \alpha^2)' = 0^{(2)}\lambda + \alpha^{(2)}$. □

We turn to the concept of full polynomials [2] and define the derivative of a polynomial f as the derivative we defined above of the full presentation of f. The full presentation of a polynomial includes inessential monomials whose derivatives are likely to match the extra monomials we obtained in our early calculation.

We turn again to prove that $[f(g(\lambda))]' = g'(\lambda) \cdot f'(g(\lambda))$. Assume that $(fg)' = f'g + g'f$ for every polynomials f and g. As shown above we can also assume that $f = \lambda^k$ and therefore $f' = 0^{(k)}\lambda^{k-1}$. We know that $f(g) = g^k = gg^{k-1}$ so $[f(g)]' = g'g^{k-1} + g(g^{k-1})' = g'g^{k-1} + g[gg^{k-2}]'$. Continuing this action we finally get k copies of $g'g^{k-1}$ which is exactly $0^{(k)}g'g^{k-1} = g'\dot{0}^{(k)}g^{k-1} = g'f'(g)$.

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