Problem assignment 2.

Algebraic Geometry and Commutative Algebra

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Remark. In this assignment you can freely use the Nullstellensatz and Serre's lemma but you should specify when you use them. For an algebraic variety X we denote by $\mathcal{O}(X)$ the algebra of global regular functions on X.

[P] 1. Let V be a finite dimensional vector space over k. Fix a natural number l and consider the set $GR_l(V)$ of all l-dimensional subspaces $L \subset V$.

Show how to describe a structure of algebraic variety on this set (this variety is called the **Gassmannian** of l-dimensional subspaces of V).

Hint. Consider the standard l-dimensional vector space $L = k^l$ and denote by H the vector space $Hom(L, V)$. Introduce on H the structure of an algebraic variety. Show that the subset $St(V) \subset H$ consisting of all imbeddings is an open subvariety in H (it is called the Stiffel variety).

Show that the group $G = GL(l, k)$ acts naturally on $St(V)$ and the quotient set is naturally identified with $Gr_l(V)$. Use the natural projection $p : St(V) \to Gr_l(V)$ to introduce the structure of a space with functions on $Gr_l(V)$.

[P] 2. Let X be an algebraic variety obtained from projective plane $\mathbf{P}^2$ by removing one point. Describe the algebra $\mathcal{O}(X)$ of global regular functions on X.

Definition. (Top) Let X be a topological space. A subset $Y \subset X$ is called **locally closed** if it satisfies the following equivalent conditions

(i) Y is an intersection of an open and a closed subsets of X.

(ii) Y is locally closed, i.e. any point $x \in Y$ has an open neighborhood U such that $U \cap Y$ is closed in U .

(iii) Y is open in $cl(Y)$ (here $cl(A)$ is the closure of A).

Check that these conditions are equivalent.

3. Let X be an algebraic variety and $Z \subset X$ a locally closed subset. Show that Z has a canonical structure of an algebraic variety.

Here you should use the following finiteness lemma that we will prove soon

Lemma. Let X be an affine algebraic variety and let U be any open subset of X. Then U can be covered by a finite number of basic open subsets X_f.

[P] 4. Let V be a finite dimensional vector space over k and $X = \mathbf{P}(V)$ the corresponding projective space. We would like to describe the algebraic structure of this space in more detail. Later we will use this description many times.

Denote by A the algebra of polynomial functions on V. This is a graded algebra $A = \oplus A^k$.

Given a homogeneous polynomial $f \in A^k$ with $k > 0$ we consider the corresponding basic open subset $V_f \subset \mathbf{V}$ and denote by X_f its image in the projective space X.

(i) Show that the sets X_f form a basis of the Zariski topology on X.

(ii) Show that every subset X_f is an affine algebraic variety and the algebra of regular functions $\mathcal{O}(X_f)$ is isomorphic to the subalgebra A^0_f of functions of degree 0 in the graded algebra A_f.

In order to do this you will need a result from linear algebra described in the next problem.

Definition. (LA) Let H be a group. By definition a **character** of H is a homomorphism $\chi : H \to k^*$.

Suppose we fixed an action of the group H on a k-vector space L. For any character χ of H we consider eigen subspace

$L^\chi := \{v \in L| \ hv = \chi(h)v \text{ for all } h \in H\}$

[P] 5. (LA). Let a group H act on a space L. Fix a collection of characters $\chi_1, ..., \chi_m$ of H pairwise distinct.

Suppose we have an equality in L of the form $v = \sum_i v_i$, where $v_i \in L^{\chi_i}$. Show that every vector v_i can be written as a linear combination of vectors hv for some elements $h \in H$.

In particular show that if $v = 0$ then all vectors v_i are 0.

∇ **6.** Let A be a finitely generated k-algebra (commutative, with 1). Consider the set $M(A) := \mathrm{Mor}_{k-alg}(A, k)$.

(i) Describe Zariski topology on the set $M(A)$. Show that the set $M(A)$ has a natural structure of an affine algebraic variety. In particular describe the algebra of regular functions $\mathcal{O}(M(A))$.

(ii) Given a morphism of k-algebras $\mu : B \to A$ show that the corresponding map $\nu : M(A) \to M(B)$ is a morphism of affine algebraic varieties.

Give an example of different morphisms of algebras that give the same morphism of varieties.

7. Show that the category Alg_k of algebraic varieties over the field k has product.

8. Let X be an algebraic variety. Consider the diagonal morphism $\Delta : X \to X \times X$.

(i) Show that the image $Z = \Delta(X)$ is a locally closed subvariety and that Δ defines an isomorphism of algebraic varieties $\Delta : X \to Z$.

(ii) Variety X is called **separated** if subset $\Delta(X)$ is closed in $X \times X$. Show that this is equivalent to the following condition

(Sep) If $U, V \subset X$ are open affine subsets, then the subset $W = U \bigcap V$ is also an open affine subset and the algebra $O(W)$ is generated by images of algebras $O(U)$ and $O(V)$.

9. (i) Consider an action ρ of the group G_m on the variety $X = \mathbf{A}^2 \setminus 0$ given by $\rho(a)(x, y) = (ax, ay)$.

Show that the variety $Y = X/G_M$ is glued from two copies of the affine line $\mathbf{A}^1$. Describe the gluing explicitly in coordinates. Show that the variety Y is separated.

(ii) Consider an action η of the group G_m on the variety $X = \mathbf{A}^2 \setminus 0$ given by $\eta(a)(x, y) = (ax, a^{-1}y)$.

Show that the variety $Z = X/G_M$ is glued from two copies of the affine line $\mathbf{A}^1$. Describe the gluing explicitly in coordinates. Show that the variety Z is not separated.

10. Let X be a separated algebraic variety. Prove the following property.

Uniqueness of extensions of morphisms. Let Z be an algebraic variety and $U \subset Z$ an open dense subset (dense means that closure of U is Z). Suppose we have a morphism of algebraic varieties $\mu : U \to X$ and we would like to extend it to a morphism $\nu : Z \to X$. Then the morphism ν is uniquely defined (if it exists).

Show by example that this is not always true if X is not separated.