



תרגיל 4 – פתרון

.1

$$\int \frac{x + \sqrt[3]{x^2} + \sqrt[6]{x}}{x(1 + \sqrt[3]{x})} dx$$

$$\left\{ \begin{array}{l} x = t^6 \\ dx = 6t^5 dt \end{array} \right.$$

$$= \int \frac{t^6 + t^4 + t}{t^6(1+t^2)} \cdot 6t^5 dt = 6 \int \frac{t^5 + t^3 + 1}{(1+t^2)} dt = 6 \int t^3 dt + 6 \int \frac{dt}{t^2+1}$$

$$= \frac{3}{2} t^4 + 6 \arctan t + c = \frac{3}{2} x^{2/3} + 6 \arctan x^{1/6} + c$$

.2

$$\int \frac{2}{(2-x)^2} \sqrt[3]{\frac{2-x}{2+x}} dx$$

$$\frac{2-x}{2+x} = t^3$$

$$2-x = (2+x)t^3$$

$$2-2t^3 = x(t^3+1)$$

$$x = \frac{2-2t^3}{t^3+1} = 2 \cdot \frac{1-t^3}{t^3+1}$$

$$dx = 2 \cdot \frac{(-3t^2(t^3+1) - 3t^2(1-t^3))}{(t^3+1)^2} dt = \frac{-12t^2}{(t^3+1)^2} dt$$

$$2-x = 2 - \frac{2(1-t^3)}{t^3+1} = \frac{4t^3}{t^3+1}$$

$$= \int \frac{2(t^3+1)^2}{16t^6} \cdot t \cdot \left(\frac{-12t^2}{(t^3+1)^2} \right) dt$$

$$= \int \frac{-24t^3}{16t^6} dt = -\frac{3}{2} \int \frac{dt}{t^3} = \frac{3}{4} \frac{1}{t^2} + c = \frac{3}{4} \left(\frac{2+x}{2-x} \right)^{2/3} + c$$



.3

$$\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{dx}{(x-1)(x+1)\sqrt[3]{\frac{x-1}{x+1}}}$$

$$\frac{x-1}{x+1} = t^3$$

$$x-1 = t^3(x+1)$$

$$x(1-t^3) = t^3 + 1$$

$$x = \frac{t^3 + 1}{1-t^3}$$

$$dx = \frac{3t^2(1-t^3) + 3t^2(t^3+1)}{(1-t^3)^2} dt = \frac{6t^2}{(1-t^3)^2} dt$$

$$x-1 = \frac{t^3+1}{1-t^3} - 1 = \frac{t^3+1-1+t^3}{1-t^3} = \frac{2t^3}{1-t^3}$$

$$x+1 = \frac{t^3+1}{1-t^3} + 1 = \frac{t^3+1+1-t^3}{1-t^3} = \frac{2}{1-t^3}$$

$$\Rightarrow \int \frac{6t^2 \cdot (1-t^3)^2}{(1-t^3)^2 4t^3 \cdot t} dt = \frac{3}{2} \int \frac{dt}{t^2} = -\frac{3}{2} \frac{1}{t} + c = -\frac{3}{2} \sqrt[3]{\frac{x+1}{x-1}} + c$$

.4

$$\int \frac{dx}{x + \sqrt{x^2 - x + 1}}$$

$$\sqrt{x^2 - x + 1} = t - x$$

$$x^2 - x + 1 = t^2 - 2tx + x^2$$

$$t^2 - 2tx + x - 1 = 0$$

$$x(1-2t) = 1-t^2$$

$$x = \frac{1-t^2}{1-2t}$$

$$dx = \frac{-2t(1-2t) + 2(1-t^2)}{(1-2t)^2} dt = \frac{2t^2 - 2t + 2}{(1-2t)^2} dt$$



$$\Rightarrow \int \frac{2(t^2 - t + 1)}{(1-2t)^2 \cdot t} dt$$

$$\frac{2(t^2 - t + 1)}{(1-2t)^2 \cdot t} = \frac{A}{t} + \frac{B}{1-2t} + \frac{D}{(1-2t)^2}$$

$$A(1-2t)^2 + Bt(1-2t) + Dt = 2t^2 - 2t + 2$$

$$t^2 : 4A - 2B = 2$$

$$t : -4A + B + D = -2$$

$$t^0 : A = 2$$

$$A = 2$$

$$B = 3$$

$$D = 3$$

$$\Rightarrow \int \frac{2dt}{t} + \int \frac{3dt}{1-2t} + \int \frac{3dt}{(1-2t)^2}$$

$$= 2 \ln|t| - \frac{3}{2} \ln|1-2t| + \frac{3}{2} \frac{1}{1-2t} + c$$

$$= 2 \ln|x + \sqrt{x^2 - x + 1}| - \frac{3}{2} \ln|1 - 2x - 2\sqrt{x^2 - x + 1}| + \frac{3}{2} \frac{1}{1 - 2x - 2\sqrt{x^2 - x + 1}} + c$$

.5

$$\int \frac{dx}{(1+x)\sqrt{1+x-x^2}}$$

$$\sqrt{1+x-x^2} = tx+1$$

$$1+x-x^2 = t^2x^2 + 2tx+1$$

$$1-x = t^2x+2t$$

$$x(t^2+1) = 1-2t$$

$$x = \frac{1-2t}{t^2+1}$$

$$dx = \frac{-2(t^2+1) - (1-2t) \cdot 2t}{(t^2+1)^2} dt = \frac{2t^2 - 2t - 2}{(t^2+1)^2} dt$$

$$1+x = 1 + \frac{1-2t}{t^2+1} = \frac{t^2+1+1-2t}{t^2+1} = \frac{t^2-2t+2}{t^2+1}$$



$$\begin{aligned} &\Rightarrow \int \frac{(2t^2 - 2t - 2) \cdot (t^2 + 1)}{(t^2 + 1)^2 \cdot (t^2 - 2t + 2) \left[\frac{t(1-2t)}{t^2 + 1} \right]} dt = \int \frac{2t^2 - 2t - 2}{(t^2 - 2t + 2)(t - 2t^2 + t^2 + 1)} dt \\ &= -2 \int \frac{(t^2 - t - 1)}{(t^2 - 2t + 2)(t^2 - t - 1)} dt = -2 \int \frac{dt}{t^2 - 2t + 2} = -2 \int \frac{dt}{(t-1)^2 + 1} = -2 \arctan(t-1) + c \\ &= -2 \arctan\left(\frac{\sqrt{1+x-x^2}-1}{x}\right) + c \end{aligned}$$

.6

$$\begin{aligned} &\int \frac{xdx}{\sqrt{(7x-10-x^2)^3}} \\ &-x^2 + 7x - 10 = 0 \\ &x^2 - 7x + 10 = 0 \\ &x_1 = 2 \quad x_2 = 5 \\ &\sqrt{7x-10-x^2} = t(x-2) \\ &\sqrt{-(x-2)(x-5)} = t(x-2) \\ &-(x-2)(x-5) = t^2(x-2)^2 \\ &-(x-5) = t^2(x-2) \\ &x(t^2+1) = 5+2t^2 \\ &x = \frac{5+2t^2}{t^2+1} \\ &dx = \frac{4t(t^2+1) - (5+2t^2) \cdot 2t}{(t^2+1)^2} dt = \frac{-6t}{(t^2+1)^2} dt \end{aligned}$$



$$\begin{aligned} &\Rightarrow \int \frac{5+2t^2}{(t^2+1)t^3 \left[\frac{5+2t^2}{t^2+1} - 2 \right]^3} \cdot \frac{(-6t)}{(t^2+1)^2} dt \\ &= -6 \int \frac{(5+2t^2) dt}{t^2(5+2t^2-2t^2-2)^3} dt = -\frac{2}{9} \int \frac{5+2t^2}{t^2} dt = \frac{10}{9} \frac{1}{t} - \frac{4}{9} t + c \\ &= \frac{10}{9} \cdot \frac{(x-2)}{\sqrt{7x-10-x^2}} - \frac{4}{9} \frac{\sqrt{7x-10-x^2}}{x-2} + c \end{aligned}$$

.7

$$\begin{aligned} &\int \frac{(x+\sqrt{1+x^2})^{15}}{\sqrt{1+x^2}} dx \\ &\sqrt{1+x^2} = t - x \\ &1+x^2 = t^2 - 2tx + x^2 \\ &1 = t^2 - 2tx \\ &x = \frac{t^2-1}{2t} \\ &dx = \frac{4t^2 - 2(t^2-1)}{4t^2} dt = \frac{2t^2+2}{4t^2} dt = \frac{t^2+1}{2t^2} dt \\ &\Rightarrow \int \frac{t^{15}}{t - \frac{t^2-1}{2t}} \cdot \frac{t^2+1}{2t^2} dt = \int \frac{t^{15}(t^2+1) \cdot 2t}{(t^2+1)2t^2} dt \\ &= \int t^{14} dt = \frac{t^{15}}{15} + c = \frac{(x+\sqrt{1+x^2})^{15}}{15} + c \end{aligned}$$