Exercise 1 in Random Walks. (Can give up to 35 points)

Note this exe contains 100 possible points, solve as many as you want, but signal up to 42 points worth of exe. That I should grade, and they can give a score of up to 35.

I expect there to be about 4 exercise pages throughout the course, though some will have a bit less variety to choose from, so you should try to hand in at least 25 points worth of exe.

(Don't worry about this now)

By graph we always mean undirected connected and finite degree unless otherwise stated. Usually we also mean uniformly bounded degrees.

Part a: RW (without need of electrical networks, though you can use them)

(Questions 1,2 can be done straightforward, or using Markov Chain theorems on the stationary distribution. 3,4 quite basic)

1. (8) Let $G=(V,E)$ be a finite d-regular graph. Let $\{X_i\}$ be a Simple random walk on G, and recall P_x denotes the distribution of the random walk started from $X_0=x$.
   Let $x,y\in V$, and let τ be the first (positive) hitting time of $\{x,y\}$ (i.e. the first time the walk hits either x or y)
   a. Show that $P_x(X_\tau = y) = P_y(X_\tau = x)$
   b. Use this to prove that E_x(#visits to y before returning to x) = 1 for every $y\in V$
   c. What happens in non-regular bounded degree graphs?
2. (8) a. Show the above result true also for infinite recurrent graphs?
   b. What about transient graphs?

c. Use (a) to show that the expected return time to a vertex in any infinite bounded degree graph is infinite.

3. (4) Prove that the probability of a random walk staying for n steps in a fixed finite subset of a connected graph decreases exponentially in n.
4. (4) Show that the measure pi(v)~deg(v) is stationary for SRW on any graph with finite degrees.

5. (*8) Find a rooted tree with exponential growth for which SRW is recurrent. (E.g. a tree s.t. |{v: d(v,o)<=n}|~2^n , where ~ is up to a mult. constant, but is recurrent) .

Part b: electrical networks

6. (6) Let H_n={0,1}^n be the Hypercube of dimension n. That is the graph whose vertices are n-tuples of bits, and two vertices are connected if they differ in just one place. Find the resistance between the (0,0,…,0) and (1,1,…,1). (Its enough to find the leading term)
   What does this tell you about the probability to reach (1,1,…,1) before returning to (0,0,…0) as n->infinity?

7. (8) Find an example of a recurrent graph not satisfying Nash-Williams. Hint: - try to construct a graph where cut-sets must grow fast , but is still recurrent. If this is hard, try at least to show nested cut sets are not enough. (Hint in white)

8. (8) Prove that the Shorting law (That if you short vertices in an electrical network, the effective resistances only decrease) and the cutting law (That if you remove an edge from the network – the effective resistances only grow) are each equivalent to Reiylighs Monotonicity principle (that increasing resistances of edges only increases effective resistances) (8)

9. (6) A random walk moves on the non-negative integers; when it is in state n, n > 0, it moves with probability p_n to n + 1 and with probability 1-p_n, to n-1. When at 0, it moves to 1. Determine

a network that gives this random walk and give a criterion in terms of the p_n for recurrence of the random walk.

10. (8) Let T_d (d>=2, or even just d=2) be the d+1 regular rooted tree (i.e. the root has d children and each vertex has one parent and d children of each own ans so on, viewed as an undirected graph). View each edge as having resistance 1. (8)
    a. What is the resistance between the root and infinity?
    b. What is the resistance between the root and the n-th level of the tree (the set of all vertices at distance n from the root)
    c. What is the resistance between the root and a specific vertex at distance n from the root.
    d. Construct a finite energy flow from the root to infinity.

11. (10 if you solve on your own, 6 if you consult online resources) The famous infinite grid question: You have an infinite Grid Z^2 of resistors with a resistance of 1 for each edge. What is the effective resistance between 2 neighbours (Say a=(0,0) and z=(1,0)). (Note you may assume no current can flow through infinity) .
    If you already know this one, try to think of Res((0,0),(1,1)) .

PeresLyons exe 2.21, 2.29 (*) , 2.31 (hint – use description of current or prop 2.2) , 2.37(general MC), 2.58 (*) , 2.72 (6 points each)

Further hints, if necessary, will be given on demand ☺