סעיף

1\.

(א)

$$\mathcal{L}\{e^{at}f(t)\}(s) = \int_0^\infty e^{at}f(t)e^{-st}\,dt = \int_0^\infty f(t)\cdot e^{-st+at}\,dt =$$

$$= \int_0^\infty f(t)\cdot e^{-(s-a)t}\,dt = \mathcal{L}\{f(t)\}(s-a) = \boxed{F(s-a)}$$

(ב)

$$\mathcal{L}\{f(at)\}(s) = \int_0^\infty f(at)\cdot e^{-st}\,dt \;=$$

נחליף משתנים
$$\begin{bmatrix} \tilde{t} = at \\ d\tilde{t} = a\,dt \end{bmatrix}$$

$$= \int_0^\infty f(\tilde{t})\cdot e^{-s\cdot\frac{\tilde{t}}{a}}\cdot\frac{d\tilde{t}}{a} = \frac{1}{a}\int_0^\infty f(\tilde{t})\cdot e^{-\frac{s}{a}\tilde{t}}\,d\tilde{t} = \frac{1}{a}\mathcal{L}\{f(t)\}\left(\frac{s}{a}\right)$$

$$= \boxed{\frac{1}{a}\cdot F\left(\frac{s}{a}\right)}$$

(ד)

$$\mathcal{L}\{t\cdot f(t)\}(s) = \int_0^\infty t f(t)\, e^{-st}\,dt = \int_0^\infty f(t)\left[-\frac{\partial}{\partial s}\cdot e^{-st}\right]dt =$$

[redacted]

$$= \int_0^\infty -\frac{\partial}{\partial s}\left[f(t)\, e^{-st}\right]dt = -\frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\,dt = -\frac{d}{ds}F(s) = \boxed{-F'(s)}$$

2

(1c) $y'' - 5y' + 4y = e^{3t} \;/\mathcal{L}\{\cdot\},\; y(0) = 2,\; y'(0) = 3$

$$s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 4Y(s) = \frac{1}{s-3}$$

$$(s^2-5s+4)Y(s) - 2s + 7 = \frac{1}{s-3}$$

$$(s^2-5s+4)Y(s) = \frac{1}{s-3} + 2s - 7 \quad /:(s^2-5s+4) = (s-1)(s-4)$$

$$Y(s) = \frac{1}{(s-1)(s-3)(s-4)} + 2\frac{s}{(s-1)(s-4)} - \frac{7}{(s-1)(s-4)} = \frac{2}{3}\cdot\frac{1}{s-4} - \frac{1}{2}\cdot\frac{1}{s-3} + \frac{11}{6}\cdot\frac{1}{s-1}$$

$$\Rightarrow y = \mathcal{L}^{-1}\{Y(s)\}(t) = \boxed{\frac{2}{3}e^{4t} - \frac{1}{2}e^{3t} + \frac{11}{6}e^{t}}$$

ז-ג'

(2) $2y'' + y' - y = e^{3t} \; / \mathcal{L}\{\cdot\} \quad y(0)=2, \; y'(0)=3$

$$2\left(s^2Y(s) - sy(0) - y'(0)\right) + sY(s) - y(0) - Y(s) = \frac{1}{s-3}$$

$$(2s^2+s-1)Y(s) - 4s - 8 = \frac{1}{s-3}$$

$$Y(s) = \frac{4s+8+\frac{1}{s-3}}{2s^2+s-1} = \frac{1}{20}\cdot\frac{1}{s-3} - \frac{5}{4}\cdot\frac{1}{s+1} + \frac{16}{5}\cdot\frac{1}{s-\frac{1}{2}} \; / \mathcal{L}^{-1}\{\cdot\}$$

$$\boxed{y(t) = \frac{1}{20}e^{3t} - \frac{5}{4}e^{-t} + \frac{16}{5}e^{\frac{1}{2}t}}$$

(2) $y'' + 2y' + y = e^{3t} \; / \mathcal{L}\{\cdot\} \quad y(0)=2, \; y'(0)=3$

$$(s+1)^2Y(s) - 2s - 5 = \frac{1}{s-3}$$

$$Y(s) = \frac{1}{16}\cdot\frac{1}{s-3} + \frac{19}{4}\cdot\frac{1}{(s+1)^2} + \frac{31}{16}\cdot\frac{1}{s+1}$$

$$\boxed{y(t) = \frac{1}{16}\cdot e^{3t} + \frac{19}{4}\cdot te^{-t} + \frac{31}{16}\cdot e^{-t}}$$

ע"פ נוסחת ההזזה

$$\mathcal{L}\{e^{at}f(t)\}(s) = F(s-a) \qquad (f(t)=t)$$

(3) $y'' + 2y' + y = e^{-t} \; / \mathcal{L}\{\cdot\} \quad y(0)=2, \; y'(0)=3$

$$(s+1)^2Y(s) - 2s - 5 = \frac{1}{s+1}$$

$$Y(s) = \frac{2}{s+1} + \frac{5}{(s+1)^2} + \frac{1}{(s+1)^3} \; / \mathcal{L}^{-1}\{\cdot\}$$

$$\boxed{y(t) = 2e^{-t} + 5te^{-t} + \frac{t^2}{2}e^{-t}} \qquad \left(f(t) = \frac{t^2}{2} \xrightarrow{\mathcal{L}} \frac{1}{s^3}\right)$$

+ נוסחת ההזזה

3-8

⓪ $y'' - y = t \,/\, \mathcal{L}\{\cdot\}$ $\quad y(0) = 2,\ y'(0) = 3$

$$(s^2 - 1)Y(s) - 2s - 3 = \frac{1}{s^2}$$

$$Y(s) = \frac{3}{s-1} - \frac{1}{s+1} - \frac{1}{s^2} \,/\, \mathcal{L}^{-1}\{\cdot\}$$

$$\boxed{y(t) = 3e^t - e^{-t} - t}$$

① $y'' + y' - 6y = e^{2t} \,/\, \mathcal{L}\{\cdot\}$ $\quad y(0) = y'(0) = 0$

$$(s^2 + s - 6)Y(s) = \frac{1}{s-2}$$

$$Y(s) = \frac{1}{25}\cdot\frac{1}{s+3} - \frac{1}{25}\cdot\frac{1}{s-2} + \frac{1}{5}\cdot\frac{1}{(s-2)^2} \,/\, \mathcal{L}^{-1}\{\cdot\}$$

$$\boxed{y(t) = \frac{1}{25}e^{-3t} - \frac{1}{25}e^{2t} + \frac{1}{5}t\cdot e^{2t}}$$

⑤ $y'' - 2y' + 7y = \sin t \,/\, \mathcal{L}\{\cdot\}$ $\quad y(0) = y'(0) = 0$

$$(s^2 - 2s + 7)Y(s) = \frac{1}{1+s^2}$$

$$Y(s) = \frac{s+3}{20(1+s^2)} - \frac{s+1}{20(s^2-2s+7)} = \frac{1}{20}\cdot\frac{s}{1+s^2} + \frac{3}{20}\cdot\frac{1}{1+s^2}$$

$$- \frac{s+1}{20\left[(s-1)^2 + \sqrt{6}^2\right]} = \frac{1}{20}\cdot\frac{s}{1+s^2} + \frac{3}{20}\cdot\frac{1}{1+s^2} - \left[\frac{1}{20}\cdot\frac{s-1}{(s-1)^2+\sqrt{6}^2} + \frac{2}{20}\cdot\frac{1}{(s-1)^2+\sqrt{6}^2}\right]$$

$$\Rightarrow \boxed{y(t) = \frac{1}{20}\cos t + \frac{3}{20}\sin t - \frac{1}{20}e^t\cos(\sqrt{6}t) - \frac{2}{20\sqrt{6}}\cdot e^t\sin(\sqrt{6}t)}$$

$$y' + y = \begin{cases} \sin t & 0 \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases} = \sin t\, H_{0,2\pi}(t) = \sin t\left(H_0(t) - H_{2\pi}(t)\right) \quad 3$$

$$y' + y = \sin t\cdot(H(t) - H(t-2\pi)) = \sin t\, H(t) - \underbrace{\sin t}_{=\sin(t-2\pi)}\, H(t-2\pi) \,/\, \mathcal{L}\{\cdot\}$$

$$\left(\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)\right)$$

$$-1 + (1+s)Y(s) = \frac{1}{1+s^2} - e^{-2\pi s}\frac{1}{1+s^2}$$

$$Y(s) = \underbrace{\frac{1}{s+1}}_{\mathrm{I}} + \underbrace{\frac{1}{(s+1)(1+s^2)}}_{\mathrm{II}} - \underbrace{e^{-2\pi s}\cdot\frac{1}{(s+1)(1+s^2)}}_{\mathrm{III}}$$

$\underline{\mathrm{I}}$: $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$\underline{\mathrm{II}}$: $\frac{1}{(s+1)(s^2+1)} = \frac{1}{2}\cdot\frac{1}{1+s} + \frac{1}{2}\cdot\frac{1}{1+s^2} - \frac{1}{2}\cdot\frac{s}{1+s^2}$

$\to \mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \frac{1}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t$

$\underline{\mathrm{III}}$: $\mathcal{L}^{-1}\left\{e^{-2\pi s}\cdot\frac{1}{(s+1)(s^2+1)}\right\} = \frac{1}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t\,\Big|_{t \mapsto t-2\pi} \cdot H(t-2\pi)$

$\to y(t) = \mathrm{I} + \mathrm{II} - \mathrm{III} =$

$\sin(t-2\pi) = \sin t$

$$= \boxed{e^{-t} + \frac{1}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t - \left(\frac{1}{2}e^{-t+2\pi} + \frac{1}{2}\sin t - \frac{1}{2}\cos t\right)\cdot H(t-2\pi)}$$

$$= \begin{cases} \frac{3}{2}e^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t & 0 \le t < 2\pi \\ \frac{3 - e^{2\pi}}{2}e^{-t} & t > 2\pi \end{cases}$$

4.

$$y''+3y'+2y=\begin{cases} t & 0\le t<1 \\ 2-t & 1\le t<2 \\ 0 & t\ge 2\end{cases}=$$

$$= t\cdot H_{0,1}(t)+(2-t)\cdot H_{1,2}(t) = t\left(H_0(t)-H_1(t)\right)+(2-t)\left(H_1(t)-H_2(t)\right)$$

$$= t\cdot H_0(t) + (2-2t)\, H_1(t) + (t-2)\, H_2(t) =$$

$$= t\cdot H(t) - 2(t-1)\, H(t-1) + (t-2)\cdot H(t-2)$$

נפעיל התמרת לפלס:

$$(s^2+3s+2)\,Y(s) = \frac{1}{s^2} - 2\cdot e^{-s}\cdot\frac{1}{s^2} + e^{-2s}\cdot\frac{1}{s^2}$$

$$Y(s) = \frac{1}{(s+1)(s+2)}\cdot\left(\frac{1}{s^2} - 2e^{-s}\cdot\frac{1}{s^2} + e^{-2s}\cdot\frac{1}{s^2}\right) =$$

$$= \underbrace{\frac{1}{2s^2} - \frac{3}{4s} + \frac{1}{1+s} - \frac{1}{4(s+2)}}_{F(s)}\cdot\left(1-2e^{-s}+e^{-2s}\right)$$

$$\Rightarrow y(t) = f(t) - 2\,H(t-1)\,f(t-1) + H(t-2)\,f(t-2)$$

כאשר

$$f(t) = -\frac{3}{4} + \frac{1}{2}t - \frac{1}{4}e^{-2t} + e^{-t}$$

ניתן להציג לחלקים ולקבל:

$$y(t)=\begin{cases} -\frac{3}{4} - \frac{e^{-2t}}{4} + e^{-t} + \frac{t}{2} & 0<t<1 \\ \frac{7}{4} + \frac{1}{2}e^{2-2t} - 2e^{1-t} - \frac{e^{-2t}}{4} + e^{-t} - \frac{t}{2} & 1<t<2 \\ \frac{1}{2}e^{2-2t} - \frac{1}{4}e^{4-2t} - 2e^{1-t} + e^{2-t} - \frac{e^{-2t}}{4} - e^{-t} & \end{cases}$$

כמובן ש $y(t)$ רציפה בתחום $\{1<x<2\}\cup\{0<x<1\}\cup\{x>2\}$

נבדוק רציפות בנקודות התפר:

$$\lim_{t\to 1^-} y(t) = \frac{4e-e^2-1}{4e^2} = \lim_{t\to 1^+} y(t)$$

$$\lim_{t\to 2^+} y(t) = \frac{3}{4} - \frac{1}{4e^4} + \frac{3}{2e^2} - \frac{2}{e} = \lim_{t\to 2^-} y(t)$$

גם עבור y' מספיק לבדוק את נקודות התפר: $(x=1,2)$

$$y'(t)=\begin{cases} \frac{1}{2}+\frac{e^{-2t}}{2}-e^{-t} & 0<t<1 \\ -\frac{1}{2}-e^{2-2t}+2e^{1-t}+\frac{e^{-2t}}{2}-e^{-t} & 1<t<2 \\ -e^{2-2t}+\frac{1}{2}e^{4-2t}+2\cdot e^{1-t}-e^{2-t}+\frac{e^{-2t}}{2}-e^{-t} & t>2 \end{cases}$$

וכן:

$$\lim_{t\to 1^-} y'(t)=\frac{1}{2}+\frac{1}{2e^2}-\frac{1}{e}=\lim_{t\to 1^+} y'(t)$$

$$\lim_{t\to 2^-} y'(t)=-\frac{1}{2}+\frac{1}{2e^4}-\frac{2}{e^2}+\frac{2}{e}=\lim_{t\to 2^+} y'(t)$$