

Inner Product Space

Achiya Bar-On

Inner Product Space

Def.: *Inner Product Space* is a vector space V over $\mathbb{F} = \mathbb{C} \setminus \mathbb{R}$ with an *inner product*, i.e.:

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$$

that satisfies

- ▶ Linearity in the first argument:
 - ▶ $\langle \alpha v_1, v_2 \rangle = \alpha \langle v_1, v_2 \rangle$
 - ▶ $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$
- ▶ Conjugate symmetry:
 - ▶ $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$
- ▶ Positive-definiteness:
 - ▶ $\langle v, v \rangle \geq 0$
 - ▶ $\langle v, v \rangle = 0 \iff v = 0$

Elementary properties

- ▶ Almost Linearity in the second argument:
 - ▶ $\langle v_1, \alpha v_2 \rangle = \bar{\alpha} \langle v_1, v_2 \rangle$
 - ▶ $\langle v_1, v_2 + v_3 \rangle = \langle v_1, v_2 \rangle + \langle v_1, v_3 \rangle$
- ▶ Generalization:
 - ▶ $\left\langle \sum_{i=1}^n \alpha_i v_i, \sum_{j=1}^m \alpha'_j v'_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \bar{\alpha}'_j \langle v_i, v'_j \rangle$
- ▶ Zero vector
 - ▶ $\langle v, 0 \rangle = \langle 0, v \rangle = 0$

Examples

- ▶ $V = \mathbb{R}^n$ with

$$\langle x, y \rangle = \left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x^t y = \sum_{i=1}^n x_i y_i$$

- ▶ $V = \mathbb{C}^n$ with

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Ex. $V = \mathbb{C}^3$, $\langle z, w \rangle = z^t \bar{w}$

Compute inner product of $v = \begin{pmatrix} 1 \\ i \\ 2 \end{pmatrix}$ with:

▶ $v_1 = (-i, -4, \sqrt{2})$

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Sol.

$$\langle v, v_4 \rangle = \langle v, v_1 \rangle + 2 \langle v, v_2 \rangle - 3 \langle v, v_3 \rangle = 2\sqrt{2} - 3i - 3 \cdot 6$$

Examples (Cont.)

- ▶ $V = \mathbb{R}^{n \times n}$ with
 $\langle A, B \rangle = \text{trace}(AB^t)$
- ▶ $V = \{f : [-1, 1] \rightarrow \mathbb{C} \mid f \text{ is continuous function} \}$ with
$$\langle f, g \rangle := \int_{-1}^1 f(x) \overline{g(x)} dx$$

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$$\langle f, g \rangle := \int_{-1}^1 f(x) \overline{g(x)} dx$$
 - ▶ Compute $\langle \sin(x), \cos(x) \rangle$
 - ▶ $\langle \sin(x), \cos(x) \rangle = \int_{-1}^1 \sin(x) \cos(x) dx = \int_{-1}^1 \frac{\sin(2x)}{2} dx = 0$

The norm induced by an inner product

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Facts:

- ▶ $\|v\| \geq 0$ and $v = 0 \iff \|v\| = 0$
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Ex:

- ▶ $V = \mathbb{C}^n$ with $\langle z, w \rangle = z^t \bar{w}$
- ▶ $\|z\| = \sqrt{\sum_{i=1}^n z_i \bar{z}_i} = \sqrt{\sum_{i=1}^n |z_i|^2}$

CauchySchwarz inequality

Th.: Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Then:

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 $(a_1 + \dots + a_n)^2 \leq n(a_1^2 + \dots + a_n^2)$

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Sol: Look at $V = \mathbb{R}^n$ with $\langle x, y \rangle = x^t y$. Define

$x = (a_1, \dots, a_n)$, $y = (1, \dots, 1)$. By CauchySchwarz inequality

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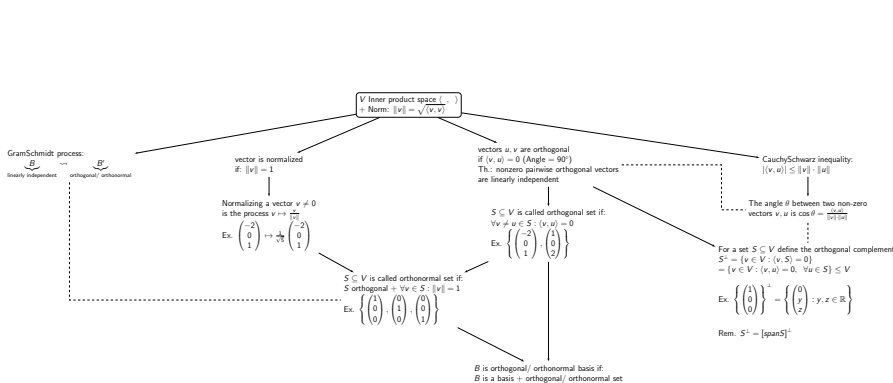
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$$|(a_1 + \dots + a_n)^2| \leq n(a_1^2 + \dots + a_n^2)$$



Th. If $0 \notin S = \{v_1, \dots, v_n\}$ orthogonal, then S linear independent.

Prof: Assume $\sum_{i=1}^n \alpha_i v_i = 0$.

Fix $1 \leq j \leq n$ and

$$0 = \langle 0, v_j \rangle = \left\langle \sum_{i=1}^n \alpha_i v_i, v_j \right\rangle = \sum_{i=1}^n \alpha_i \langle v_i, v_j \rangle = \alpha_j \langle v_j, v_j \rangle$$

Since $v_j \neq 0$, then $\alpha_j = 0$

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Ex. Let $v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3$, define $S = \{u, v\}$.

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Find basis to S^\perp .

Sol. If $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$ then $S^\perp = N(A)$. Therefore

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$$N(A) = \left\{ \begin{pmatrix} -t \\ t \\ 0 \end{pmatrix} \right\} : t \in \mathbb{R}.$$

$$\text{And basis } \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Gram-Schmidt process

Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| * \|$.

GramSchmidt process:

- ▶ Input: Linearly independent set $B = \{v_1, \dots, v_n\}$
- ▶ Output: Orthogonal/Orthonormal set $B' = \{w_1, \dots, w_n\}$
s.t. $\text{span}(B) = \text{span}(B')$

Gram-Schmidt process $B \rightsquigarrow B'$

$$B = \{v_1, \dots, v_n\}, B' = \{w_1, \dots, w_n\}.$$

The Algo.

$$w_1 := v_1$$

$$w_2 := v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$w_3 := v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$$

\vdots

$$w_i := v_i - \sum_{k=1}^{i-1} \frac{\langle v_i, w_k \rangle}{\|w_k\|^2} w_k$$

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B' Orthogonal. Normalized each w_i implies B' Orthonormal.

Example

$B = \{v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\}$. Perform GramSchmidt, to obtain an orthogonal set of vectors

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Example

$$\begin{aligned}w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2 \\&= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{\left(\frac{1}{2}\right)}{\left(\frac{6}{4}\right)} \cdot \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\&= \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \\&= \frac{1}{6} \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}\end{aligned}$$

Example

Now, $\{w_1, w_2, w_3\}$ Orthogonal.

and $\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ Orthonormal

Corollaries and Properties

- ▶ It is possible to start with $w_1 = v_1$
- ▶ Every $W \leq V$ has an orthonormal basis
- ▶ For each $1 \leq t \leq n$:
$$\text{span}(\{v_1, \dots, v_t\}) = \text{span}(\{w_1, \dots, w_t\})$$