Problem assignment 1.

Algebraic Geometry and Commutative Algebra

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Remark. In this assignment you can freely use the Nullstellensatz and Serre's lemma, but you should specify when you use them. For an algebraic variety X we denote by $\mathcal{O}(X)$ the algebra of global regular functions on X.

1. (i) Let A be a finitely generated k-algebra with no nilpotents. Consider the set $X = Mor_{k-alg}(A, k)$.

Show that A embeds into the algebra $\mathcal{F}(X)$ of k-valued functions on X and that the pair (X, A) is an affine algebraic variety.

(ii) If X, Y are affine algebraic varieties, show that
$Mor(X, Y) = Mor_{k-alg}(\mathcal{P}(Y), \mathcal{P}(X))$.

Definition. (CA) Let A be a ring (commutative, with 1). Given an element $f \in A$ we define the **localization** of A with respect to f to be the ring $A_f = A[t]/(ft-1)A[t]$.

2. (CA) (i) Show that we can define the ring A_f as a set of fractions $\frac{a}{f^k}$ modulo the following equivalence relation: $\frac{a}{f^k}$ is equivalent to $\frac{b}{f^l}$ iff for some N we have $f^N f^l a = f^N f^k b$

(ii) Describe the kernel of the canonical morphism $i : A \to A_f$.

In what cases the localized ring A_f is trivial, i.e. consists of one element 0 ?

(iii) Show that if A has no nilpotents (resp. no zero divisors), then A_f has no nilpotents (resp. no zero divisors).

[P] 3. Let $(X, \mathcal{P}(X))$ be an affine algebraic variety. Pick an element f in $\mathcal{P}(X)$. Show that the algebra $\mathcal{P}(X)_f$ is canonically isomorphic to the algebra of polynomial functions on the basic open subset X_f.

[P] 4. Let X be an affine algebraic variety and B a subset of X. Suppose we know that B is a basic open subset, i.e. it has a form $B = X_f$ for some function $f \in \mathcal{P}(X)$.

Show how to describe the subalgebra $\mathcal{P}(B) \subset \mathcal{F}(B)$ in terms of the set B, without knowing the function f that defines it.

5. Consider the subvariety $Y \subset \mathbf{A}^2$ defined by one equation $x^2 - y^3 = 0$.

Describe a morphism $\nu : \mathbf{A}^1 \to Y$ which is bijective. Show that it is a homeomorphism, but not an isomorphism of algebraic varieties.

[P] 6. Let K be some field. Consider a system of polynomial equations $\Xi = (P_\alpha)$, where P_α are polynomials in variables $x_1, ..., x_n$.

Suppose we know that there exists a field extension Ω such that the system of equations Ξ has some solution in Ω, i.e. there exists a collection of elements $x_i \in \Omega$ that satisfy all the equations in Ξ.

Show that the system Ξ has a solution in the field $k = \bar{K}$ - algebraic closure of K.

7. Let X be an affine algebraic variety. Denote by $\mathcal{B}$ the family of basic open subsets of X.

Show that $\mathcal{B}$ forms a base of a topology and that the topology defined by $\mathcal{B}$ is the Zariski topology on X.

Definition. A topological space X is called **quasicompact** if any open covering $\{U_\alpha\}$ of X has finite subcovering.

[P] 8. Show that any affine algebraic variety is quasicompact in Zariski topology.

Definition. An **algebraic variety** is a space with a sheaf of functions $(X, \mathcal{T}(X), \mathcal{O}(X))$ that satisfies the following condition

(*) There exists a finite covering of X by open subsets U_i such that for every i the space with a sheaf of functions $\mathbf{U}_i$ obtained by restriction of structures to set U_i is isomorphic to some affine algebraic variety.

Intuitively this means that X is glued from affine algebraic varieties U_i.

[P] 9. Let X be an algebraic variety. Fix a finite open covering of X consisting of affine algebraic varieties B_i.

Show that a function ϕ on X is regular iff its restriction to every subset B_i is polynomial.

Show how using the covering (B_i) one can give an explicit description of the space of regular functions $\mathcal{O}(X)$ as the kernel of a morphism $\nu : \oplus_i \mathcal{P}(B_i) \to \oplus_{i,j} \mathcal{P}(B_i \bigcap B_j)$ (here we assume that all the intersections $B_i \bigcap B_j$ are affine - this will be true for most of interesting cases).

[P] 1.0 Fix a finite dimensional vector space V over k.

(i) Describe the corresponding algebraic variety (affine space) $\mathbf{V}$ whose set of points is V. Describe explicitly the algebra $\mathcal{O}(\mathbf{V})$.

(ii) Let f be a coordinate function on V (i.e. a non-zero linear function). Consider basic open subvariety $\mathbf{V}_f \subset \mathbf{V}$. Show that V_f is an affine algebraic variety and that it is isomorphic to $H \times k^*$, where H is the hyperplane given by equation $f(x) = 1$. Describe explicitly $\mathcal{O}(\mathbf{V}_f)$.

(iii) Denote by $\mathbf{V}^*$ an open algebraic subvariety $\mathbf{V}^* = \mathbf{V} \setminus 0 \subset \mathbf{V}$. Describe the algebra $\mathcal{O}(\mathbf{V}^*)$.

Hint. Do this first for the case $n = 1$, then $n = 2$, then the general case.

The quotient construction. Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a space with a sheaf of functions. Let $p : X \to Y$ be an epimorphic map of sets. Show how in this case one can canonically define on Y the structure of a space with sheaf of functions.