Limits

```
reset():
```

```
limit((1 - cos(x))/sin(x)^2, x)
```

$$\frac{1}{2}$$

```
limit((1 + 1/n)^n, n = infinity)
```

$$\mathrm{e}$$

```
limit(1/sin(x), x = 0);
limit(1/x, x = 0, Left);
limit(1/x, x = 0, Right)
```

$$\mathrm{undefined}$$

$$-\infty$$

$$\infty$$

The function $\sin(x)$ oscillates for $x \to \infty$ between -1 and 1; no accumulation points outside that interval exist:

```
limit(sin(x), x = infinity, Intervals)
```

$$[-1, 1]$$

`limit` is not able to compute the limit of x^n for $x \to \infty$ without additional information about the parameter n:

```
assume(n in Z_):
limit(sin(x^n), x = infinity)
```

$$\begin{cases} \sin(1) & \text{if } n = 0 \\ 0 & \text{if } n \le -1 \end{cases}$$

We can also assume immediately that $n > 0$ and get no case analysis then:

```
assume(n > 0):
limit(sin(x^n), x = infinity)
```

$$\mathrm{undefined}$$

Similarly, we can assume that n=0:

```
assume(n = 0):
limit(sin(x^n), x = infinity)
```

$$\sin(1)$$

Compute limit of the piecewise function:

```
f:=piecewise([x^3 > 10000*x, 1/x], [x^3 <= 10000*x, 10])
```

$$\begin{cases} \frac{1}{x} & \text{if } 10000\,x < x^3 \\ 10 & \text{if } x^3 \le 10000\,x \end{cases}$$

```
limit(f, x = 100, Left);
limit(f, x = 100, Right);
```

$$10$$

$$\frac{1}{100}$$

```
limit(f, x = -10);
limit(f, x = 1);
```

$$-\frac{1}{10}$$

$$10$$

Derivatives

```
reset():
```

You can differentiate with respect to more than one variable with a single `diff` call. In the following example, we differentiate first with respect to `x` and then with respect to `y`:

```
diff(x^2*sin(y), x, y) = diff(diff(x^2*sin(y), x), y)
```

$$2\,x\cos(y) = 2\,x\cos(y)$$

We use the sequence operator $ to compute the third derivative of the following expression with respect to `x`:

```
diff(sin(x)*cos(x), x $ 3)
```

$$4\sin(x)^2 - 4\cos(x)^2$$

`diff` knows how to differentiate symbolic integrals:

```
int(f(x), x);
diff(%, x, x);
```

$$\int f(x)\,\mathrm{d}x$$

$$\frac{\partial}{\partial x} f(x)$$

```
g:=int(f(t, x), t = x..x^2);
diff(g, x);
```

$$\int_x^{x^2} f(t, x)\,\mathrm{d}t$$

$$\int_x^{x^2} \frac{\partial}{\partial x} f(t, x)\,\mathrm{d}t - f(x, x) + 2\,x\,f(x^2, x)$$

`diff` knows how to differentiate piecewise functions:

```
f:=piecewise([x^3 > 10000*x, 1/x], [x^3 <= 10000*x, 10]);
diff(f,x);
```

$$\begin{cases} \frac{1}{x} & \text{if } 10000\,x < x^3 \\ 10 & \text{if } x^3 \le 10000\,x \end{cases}$$

$$\begin{cases} -\frac{1}{x^2} & \text{if } 10000\,x < x^3 \\ 0 & \text{if } x^3 < 10000\,x \end{cases}$$

Exercise

Find c_0, c_1, c_2 such as and b such as $f(x) = \begin{cases} \mathrm{e}^x & \text{if } 0 < x \\ c_2\,x^2 + c_1\,x + c_0 & \text{if } x \le 0 \end{cases}$ **belongs to** C^2

```
reset():
```

```
f1:=exp(x):
f2:=c_0+c_1*x + c_2*x^2:
sys:={f1=f2,diff(f1,x)=diff(f2,x),diff(f1,x$2)=diff(f2,x$2)};
sys1:=subs(sys,x=0);
numeric::fsolve(sys1,{c_0,c_1,c_2});
```

$$\{\mathrm{e}^x = c_1 + 2\,c_2\,x,\ \mathrm{e}^x = c_2\,x^2 + c_1\,x + c_0,\ \mathrm{e}^x = 2\,c_2\}$$

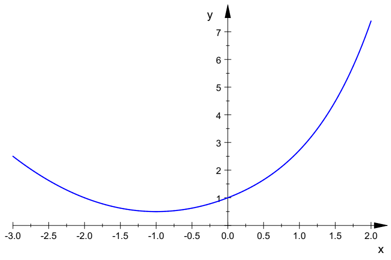
$$\{\mathrm{e}^0 = c_0,\ \mathrm{e}^0 = c_1,\ \mathrm{e}^0 = 2\,c_2\}$$

$$[c_0 = 1.0,\ c_1 = 1.0,\ c_2 = 0.5]$$

Draw function $f = \begin{cases} \mathrm{e}^x & \text{if } 0 < x \\ \frac{x^2}{2} + x + 1 & \text{if } x \le 0 \end{cases}$ **for -3<=x<=2**

```
f:=x->piecewise([x>0, exp(x)],[x<=0, 1+x+x^2/2]);
plot(f(x),x=-3..2);
```

$$x \to \mathrm{piecewise}\left([0 < x, \mathrm{e}^x], \left[x \le 0, 1 + x + \frac{x^2}{2}\right]\right)$$



Find a and b such as f is continuous and it's first derivative is continuous

```
assume(a<2);
f:=piecewise([2>x > a, (b*a^2-2*a*x-x^2)^3], [x >= 2, a*x+3*b]);
```

$$\begin{cases} 3\,b+a\,x & \text{if } 2 \le x \\ -(-b\,a^2+2\,a\,x+x^2)^3 & \text{if } a < x \wedge x < 2 \end{cases}$$

```
l_f:=limit(f,x=2,Right);
r_f:=limit(f,x=2,Left);
```

$$2\,a+3\,b$$

$$-(-b\,a^2+4\,a+4)^3$$

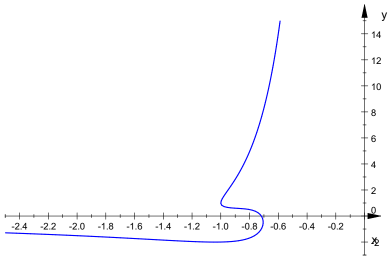
```
der_f:=diff(f,x);
l_der:=limit(der_f,x=2,Right);
r_der:=limit(der_f,x=2,Left);
```

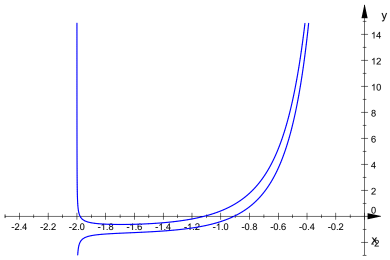
$$\begin{cases} a & \text{if } 2 < x \\ -3\,(2\,a+2\,x)\,(-b\,a^2+2\,a\,x+x^2)^2 & \text{if } a < x \wedge x < 2 \end{cases}$$

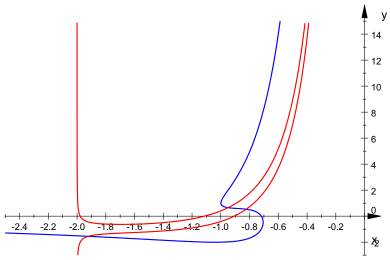
$$a$$

$$-3\,(2\,a+4)\,(-b\,a^2+4\,a+4)^2$$

```
plot(l_f=r_f,a=-2.5..0,b=-3..15);
plot(l_der=r_der,a=-2.5..0,b=-3..15);
plot(l_f=r_f,l_der=r_der,a=-2.5..0,b=-3..15);
```







```
numeric::solve([l_f=r_f,l_der=r_der],[a,b])
```

$$\{[a=-1.923027381,\ b=-1.550193212],\ [a=-0.949840617,\ b=0.6527365335],\ [a=-0.8237026521,\ b=0.53584489]\}$$

Implicite differentiation

Example

Find the slope, **m**, of the tangent line to the graph of the **cardioid** with equation:

$$x^4+2\,x^2\,y^2-4\,x^2\,y-4\,x^2+y^4-4\,y^3=0$$

at the point **P** $= \left(\frac{\sqrt{3}}{2}+1 \;,\; \sqrt{3}+\frac{3}{2} \right)$.

First, we enter the equation of the cardioid and verify that the point P lies on the curve.

```
reset();
```

```
x1:=(2+sqrt(3))/2; y1:=(3+2*sqrt(3))/2;
```

$$\frac{\sqrt{3}}{2}+1$$

$$\sqrt{3}+\frac{3}{2}$$

```
eq:=x^4+y^4-4*(y^3+x^2+(x^2)*y) + 2*(x^2)*(y^2);
```

$$x^4+2\,x^2\,y^2-4\,x^2\,y-4\,x^2+y^4-4\,y^3$$

```
eqcheck:=subs(subs(eq,x=x1),y=y1); expand(eqcheck);
```

$$\left(\sqrt{3}+\frac{3}{2}\right)^4-4\left(\sqrt{3}+\frac{3}{2}\right)^3-4\left(\frac{\sqrt{3}}{2}+1\right)^2+\left(\frac{\sqrt{3}}{2}+1\right)^4-4\left(\frac{\sqrt{3}}{2}+1\right)^2\left(\sqrt{3}+\frac{3}{2}\right)+2\left(\frac{\sqrt{3}}{2}+1\right)^2\left(\sqrt{3}+\frac{3}{2}\right)^2$$

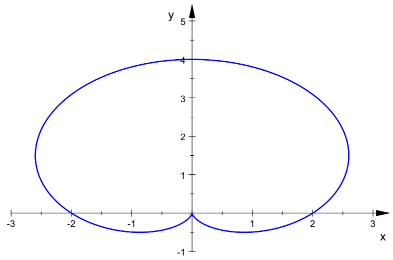
$$0$$

Next, we sketch a graph of the cardioid in the coordinate plane using the *MuPAD* command **Implicit2d** which

```
p1:=plot::Implicit2d(eq,x=-3..3,y=-1..5);
```

$$\text{plot::Implicit2d}(2\,x^2\,y^2-4\,x^2\,y-4\,x^2+x^4-4\,y^3+y^4,\ x=-3..3,\ y=-1..5)$$

```
plot(p1);
```



Now we tell *MuPAD* to treat y as a function of x :

```
y:=f(x);
```

$$f(x)$$

```
eq;
```

$$x^4+2\,x^2\,f(x)^2-4\,x^2\,f(x)-4\,x^2+f(x)^4-4\,f(x)^3$$

Observe that *MuPAD* has replaced each occurence of y by f(x).

To differentiate this equation with respect to x , we use the **diff** command.

```
deq:=diff(eq,x);
```

$$4\,x^2\,f(x)\,\frac{\partial}{\partial x}\,f(x)-4\,x^2\,\frac{\partial}{\partial x}\,f(x)+4\,f(x)^3\,\frac{\partial}{\partial x}\,f(x)-12\,f(x)^2\,\frac{\partial}{\partial x}\,f(x)+4\,x^3+4\,x\,f(x)^2-8\,x\,f(x)-8\,x$$

Now we solve the derivative of the equation for the derivative of f(x)
using the result of the implicit differention.

```
iddeq:=diff(f(x),x)=solve(deq, diff(f(x),x));
```

$$\frac{\partial}{\partial x}\,f(x)=\begin{cases}\left\{\frac{-x^3-x\,f(x)^2+2\,x\,f(x)+2\,x}{\sigma_5-x^2+f(x)^3-\sigma_2}\right\} & \text{if } \sigma_4\neq x^2+\sigma_2\\ \emptyset & \text{if } \sigma_1\neq\sigma_3\wedge x\neq 0\wedge\sigma_4=x^2+\sigma_2\\ \mathbb{C} & \text{if } (\sigma_1=\sigma_3\vee x=0)\wedge\sigma_4=x^2+\sigma_2\end{cases}$$

where

$$\sigma_1=x^2+f(x)^2$$

$$\sigma_2=3\,f(x)^2$$

$$\sigma_3=2\,f(x)+2$$

$$\sigma_4=\sigma_5+f(x)^3$$

$$\sigma_5=x^2\,f(x)$$

The point **P** has coordinates x1 and y1 , so in order to find the slope **m**
of the tangent at **P** , we must replace f(x) with y1 and x with x1.

We first get the formula expressing the derivative ***dy/dx***. Then we
substitute the appropriate values for x and for y using the **subs** command.

```
fprimex:=iddeq[2][1][1];
```

$$\frac{-x^3-x\,f(x)^2+2\,x\,f(x)+2\,x}{x^2\,f(x)-x^2+f(x)^3-3\,f(x)^2}$$

```
subs(fprimex,f(x)=y1);
```

$$-\frac{2\,x+2\,x\left(\sqrt{3}+\frac{3}{2}\right)-x^3-x\left(\sqrt{3}+\frac{3}{2}\right)^2}{3\left(\sqrt{3}+\frac{3}{2}\right)^2-\left(\sqrt{3}+\frac{3}{2}\right)^3+x^2-x^2\left(\sqrt{3}+\frac{3}{2}\right)}$$

```
subs(%,x=x1);
```

$$-\frac{2\,\sigma_2\,\sigma_1+\sqrt{3}-\sigma_2{}^3-\sigma_2\,\sigma_1{}^2+2}{3\,\sigma_1{}^2-\sigma_1{}^3+\sigma_2{}^2-\sigma_2{}^2\,\sigma_1}$$

where

$$\sigma_1=\sqrt{3}+\frac{3}{2}$$

$$\sigma_2=\frac{\sqrt{3}}{2}+1$$

The percent symbol **(%)** is a place-holder for the results of the immediately
preceding operation. Then we **simplify** our answer.

```
m:=simplify(%);
```

$$-1$$

First we remove the definition of y as f(x) with the **delete** command.

```
delete (y);
```

```
eq2:=y-m*(x-x1)-y1;
```

$$x+y-\frac{3\,\sqrt{3}}{2}-\frac{5}{2}$$

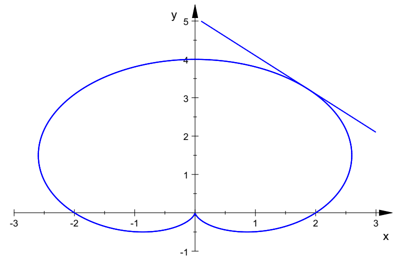
Now we make and store an implicit plot of the tangent line.

The semi-colon are suppresses the output from the command.

```
p2:=plot::Implicit2d(eq2,x=-3..3,y=-1..5):
```

Now we plot the tangent line on the same set of axes as the cardioid.

```
plot(p1,p2);
```



Taylor series

We compute a Taylor series around the **default point 0**:

```
reset():
```

```
s := taylor(exp(exp(x)), x)
```

$$e+x\,e+x^2\,e+\frac{5\,x^3\,e}{6}+\frac{5\,x^4\,e}{8}+\frac{13\,x^5\,e}{30}+O(x^6)$$

Default order of Teylor series is **6**.

```
s := taylor(exp(exp(x)), x, 15)
```

$$e+x\,e+x^2\,e+\frac{5\,x^3\,e}{6}+\frac{5\,x^4\,e}{8}+\frac{13\,x^5\,e}{30}+\frac{203\,x^6\,e}{720}+\frac{877\,x^7\,e}{5040}+\frac{23\,x^8\,e}{224}+\frac{1007\,x^9\,e}{17280}+\frac{4639\,x^{10}\,e}{145152}+\frac{22619\,x^{11}\,e}{1330560}+\frac{4213597\,x^{12}\,e}{479001600}+\frac{27644437\,x^{13}\,e}{6227020800}+\frac{95449661\,x^{14}\,e}{43589145600}+O(x^{15})$$

The result of `taylor` is of the following domain type:

```
domtype(s)
```

Series::Puiseux

If we apply the function expr to a series, we get an arithmetical expression without the order term:

```
expr(s)
```

$$\frac{95449661\,e\,x^{14}}{43589145600}+\frac{27644437\,e\,x^{13}}{6227020800}+\frac{4213597\,e\,x^{12}}{479001600}+\frac{22619\,e\,x^{11}}{1330560}+\frac{4639\,e\,x^{10}}{145152}+\frac{1007\,e\,x^9}{17280}+\frac{23\,e\,x^8}{224}+\frac{877\,e\,x^7}{5040}+\frac{203\,e\,x^6}{720}+\frac{13\,e\,x^5}{30}+\frac{5\,e\,x^4}{8}+\frac{5\,e\,x^3}{6}+e\,x^2+e\,x+e$$

```
domtype(%)
```

DOM_EXPR

```
delete s:
```

A Taylor series expansion of $g(x)=\frac{1}{\left(\frac{\sin(x)}{x}-1\right)^2}$ around x=0 does not exist. Therefore, `taylor` responds with an error message:

```
taylor(1/(1-sin(x)/x)^2, x = 0)
Error: Cannot compute a Taylor expansion of '1/(1/x*sin(x) - 1)^2'. Try 'series' for a more general expansion. [taylor]
```

Following the advice given in this error message, we try series to compute a more general series expansion. A Laurent expansion does exist:

```
series(1/(1-sin(x)/x)^2, x = 0)
```

$$\frac{36}{x^4}+\frac{18}{5\,x^2}+\frac{129}{700}+O(x^2)$$

Multivariate Taylor series

```
reset():
```

```
f:=(x,y)->cos(cos(x)*y^2+cos(y))
```

$$(x,y)\rightarrow\cos(\cos(x)\,y^2+\cos(y))$$

```
t:=mtaylor(f(x,y),[x,y],6)
```

$$\frac{\sin(1)\,x^2\,y^2}{2}+\left(-\frac{\cos(1)}{8}-\frac{\sin(1)}{24}\right)y^4-\frac{\sin(1)\,y^2}{2}+\cos(1)$$

We compute a Taylor series around the origin (default). The expansion contains all terms through total degree 3:

```
mtaylor(exp(x^2 - y), [x, y], 4)
```

$$-x^2\,y+x^2-\frac{y^3}{6}+\frac{y^2}{2}-y+1$$

We request additional terms of higher order:

```
mtaylor(exp(x^2 - y), [x, y], 5)
```

$$\frac{x^4}{2}+\frac{x^2\,y^2}{2}-x^2\,y+x^2+\frac{y^4}{24}-\frac{y^3}{6}+\frac{y^2}{2}-y+1$$

In the example above, the leading term is of total degree 0. In the following example, the leading term is of total degree 2. Thus, the default mode `RelativeOrder` produces terms of total degree smaller than 4 + 2 = 6:

```
mtaylor(x*y*exp(x^2 - y), [x, y], 4)
```

$$-x^3\,y^2+x^3\,y-\frac{x\,y^4}{6}+\frac{x\,y^3}{2}-x\,y^2+x\,y$$

We request an absolute truncation order of 4, so that only terms of total degree smaller than 4 are computed:

```
mtaylor(x*y*exp(x^2 - y), [x, y], AbsoluteOrder = 4)
```

$$x\,y-x\,y^2$$

A common problem in symbolic calculations is “expression swell:” Intermediate expressions which are not or cannot be simplified lead to unnecessarily complicated results. The following is an example of such behavior:

```
mtaylor((a+x)^n, x, 4)
```

$$\sigma_1-x^2\,\sigma_1\left(\frac{n}{2\,a^2}-\frac{n^2}{2\,a^2}\right)-x^3\,\sigma_1\left(\frac{n^2}{4\,a^3}-\frac{n}{3\,a^3}+\frac{n\left(\frac{n}{4\,a^2}-\frac{n^2}{6\,a^2}\right)}{a}\right)+\frac{n\,x\,\sigma_1}{a}$$

where

$$\sigma_1=e^{n\ln(a)}$$

In general, applying simplify or Simplify to complicated results is a strategy that often helps. In this case, however, it would destroy the format of the series:

```
simplify(%)
```

$$\frac{a^{n-3}\,(6\,a^3+6\,a^2\,n\,x+3\,a\,n^2\,x^2-3\,a\,n\,x^2+n^3\,x^3-3\,n^2\,x^3+2\,n\,x^3)}{6}$$

What is required is a way to map a function like simplify to the coefficients of the series only. Since `mtaylor` returns an ordinary expression, this must be done in the `mtaylor` call itself, using the `Mapcoeffs` option:

```
mtaylor((a+x)^n, x, 4, Mapcoeffs=simplify)
```

$$a^n+a^{n-1}\,n\,x+\frac{a^{n-2}\,n\,x^2\,(n-1)}{2}+\frac{a^{n-3}\,n\,x^3\,(n^2-3\,n+2)}{6}$$

Error finding

```
reset():
```

```
f:=cos(x*cos(y)+x*sin(y))
```

$$\cos(x\cos(y)+x\sin(y))$$

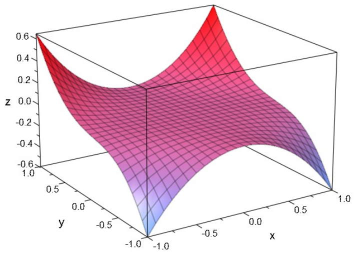
```
t1:=mtaylor(f,[x,y],5)
```

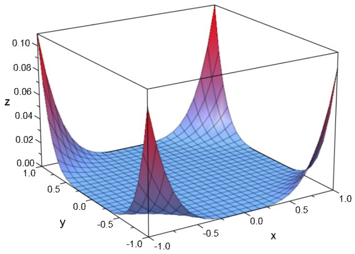
$$\frac{x^4}{24} - \frac{x^2}{2} - x^2\,y + 1$$

```
t2:=mtaylor(f,[x,y],10)
```

$$\frac{x^8\,y}{5040} + \frac{x^8}{40320} - \frac{x^6\,y^3}{180} - \frac{x^6\,y^2}{60} - \frac{x^6\,y}{120} - \frac{x^6}{720} + \frac{x^4\,y^5}{45} - \frac{2\,x^4\,y^4}{9} - \frac{x^4\,y^3}{9} + \frac{x^4\,y^2}{6} + \frac{x^4\,y}{6} + \frac{x^4}{24} + \frac{4\,x^2\,y^7}{315} - \frac{2\,x^2\,y^5}{15} + \frac{2\,x^2\,y^3}{3} - x^2\,y - \frac{x^2}{2} + 1$$

```
err1:=(f-t1):
err2:=abs(f-t2):
plotfunc3d(err1,x=-1..1,y=-1..1):
plotfunc3d(err2,x=-1..1,y=-1..1):
```





Exercise 1

מצא וצייר את הגרפים של פונקציה וטור טיילור שלה מסדר 11 ו- 15 בתלום 0.9..1.6

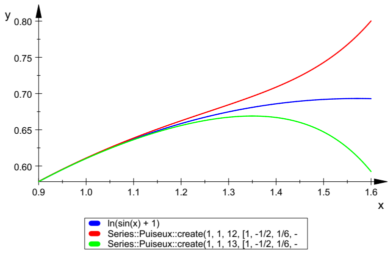
f(x)=ln(sin(x)+1)

```
f[0]:= ln(sin(x)+1);
f[11]:=taylor(f[0],x=0,11);
f[12]:=taylor(f[0],x=0,12);
plot(f[0],f[11], f[12],x=0.9..1.6,LegendVisible);
```

$$\ln(\sin(x) + 1)$$

$$x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \frac{x^6}{45} + \frac{61\,x^7}{5040} - \frac{17\,x^8}{2520} + \frac{277\,x^9}{72576} - \frac{31\,x^{10}}{14175} + \frac{50521\,x^{11}}{39916800} + O(x^{12})$$

$$x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} - \frac{x^6}{45} + \frac{61\,x^7}{5040} - \frac{17\,x^8}{2520} + \frac{277\,x^9}{72576} - \frac{31\,x^{10}}{14175} + \frac{50521\,x^{11}}{39916800} - \frac{691\,x^{12}}{935550} + O(x^{13})$$



תמצאו את הקטעים הכי גדולים בהם הטעויות בקירובים הנ"ל קטנים מ- 0.1

```
numeric::solve(abs(f[0]-expr(f[11]))=0.1,x=0..infinity);
numeric::solve(abs(f[0]-expr(f[12]))=0.1,x=0..infinity);
```

$$\{1.590227416\}$$

$$\{1.599252115\}$$

תמצאו את הסדר הקטן ביותר כך שהטעות המקסימלית בטור טיילור תהיה קטנה מ- 0.6

```
f:=ln(sin(x)+1):
f1:=expr(taylor(f,x,1)):
i:=1:
while abs(float(subs(f-f1,x=1.6)))>0.06 do
  i:=i+1;
  f1:=expr(taylor(f,x,i)):
end_while:
i;
```

$$28$$

```
x=PI+sqrt(1-y^2), -1<=y<=1
```

$$x = \pi + \sqrt{1 - y^2},\ (-1 \le y) \le 1$$

Exercise 2

תחשבו את השגיאה הגדולה ביותר של טור טיילור מסדר 10 של פונקציה

$$e^{\cos(y)\,\sin(x)}$$

מסביב ל

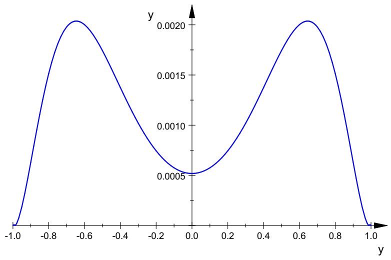
$$x = \pi,\ y = 0$$

בעקומה

$$x = \pi + \sqrt{1 - y^2},\ (-1 \le y) \le 1$$

```
reset():
f:=exp(sin(x)*cos(y)):
f1:=mtaylor(f,[x=PI,y=0],10):
x:=PI+sqrt(1-y^2):
f2:=abs(Simplify(f1-f)):
plot(f2,y=-1..1);
df2:=diff(f2,y):
y0:=numeric::solve(df2,y=0.6);
```

```
err:=float(subs(f2, y=y0[1]));
```



```
{0.6459511737}
0.00203704176
```

Exercise 3

Find asymptotic series of $\frac{1}{\sin(x)}$ **around x=0 (of order 10)**

```
reset():
```

```
series(1/sin(x),x=0,10);
```

$$\frac{1}{x}+\frac{x}{6}+\frac{7\,x^3}{360}+\frac{31\,x^5}{15120}+\frac{127\,x^7}{604800}+O(x^9)$$

Plot the graphics of asymptotic expansions of `f(x)=ln(x)/(1-x)` **around** $x = 0,\ x = \infty$ **(of order 10)**

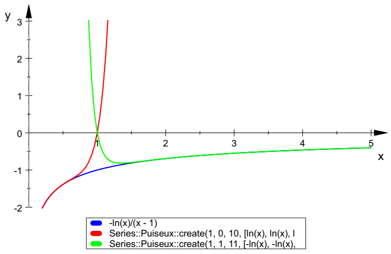
```
reset():
```

```
f:=ln(x)/(1-x);
f1:=series(f,x=0,10);
f2:=series(f,x=infinity,10);
plot(f,f1,f2,x=0..5,ViewingBoxYRange = -2..3, LegendVisible);
```

$$-\frac{\ln(x)}{x-1}$$

$$\ln(x)+x\ln(x)+x^2\ln(x)+x^3\ln(x)+x^4\ln(x)+x^5\ln(x)+x^6\ln(x)+x^7\ln(x)+x^8\ln(x)+x^9\ln(x)+O(x^{10})$$

$$-\frac{\ln(x)}{x}-\frac{\ln(x)}{x^2}-\frac{\ln(x)}{x^3}-\frac{\ln(x)}{x^4}-\frac{\ln(x)}{x^5}-\frac{\ln(x)}{x^6}-\frac{\ln(x)}{x^7}-\frac{\ln(x)}{x^8}-\frac{\ln(x)}{x^9}-\frac{\ln(x)}{x^{10}}+O\left(\frac{1}{x^{11}}\right)$$



HOME READING

Using derivatives to find absolute maxima and minima

DERIVATIVES

Differentiation is a process that, in most instances, involves only a few rules which are used over and over. Even for relatively simple functions, such as those in the examples and exercises that follow, the results may quickly become rather complicated and unwieldy. Therefore differentiation lends itself very well to execution by a computer.

If f has been entered as a function in MuPAD, then the command "D(f);" yields the derivative of *f*. For example, let

$$f(x) = x^2 \sec(x)\,.$$

We will find the first and second derivatives of *f*.

```
reset()
```

```
f:=x->x^2 * sec(x);
```

$$x \to x^2 \sec(x)$$

```
D(f);
```

$$x \to \frac{2\,x}{\cos(x)}+\frac{x^2\sin(x)}{\cos(x)^2}$$

```
D(D(f));
```

$$x \to \frac{2}{\cos(x)}+\frac{x^2}{\cos(x)}+\frac{2\,x^2\sin(x)^2}{\cos(x)^3}+\frac{4\,x\sin(x)}{\cos(x)^2}$$

To compute the n th derivative, we can use "(D@@n)(f);" thus the third derivative of the function f defined above is:

```
(D@@3)(f);
```

$$x \to \frac{6\,x}{\cos(x)}+\frac{6\sin(x)}{\cos(x)^2}+\frac{5\,x^2\sin(x)}{\cos(x)^2}+\frac{12\,x\sin(x)^2}{\cos(x)^3}+\frac{6\,x^2\sin(x)^3}{\cos(x)^4}$$

FINDING THE ABSOLUTE MAXIMUM AND MINIMUM

The theory tells us that a continuous function defined on a closed interval always has an absolute maximum M and an absolute minimum m; i.e., there are numbers and in [a,b] such that m = f(α) ≤ f(x) ≤ f(β) = M for all x in [a , b]. Moreover, to find them we need only consider the endpoints a and b and the critical points, i.e., the solutions to the equation f '(x) = 0, and values of x for which f '(x) does not exist.

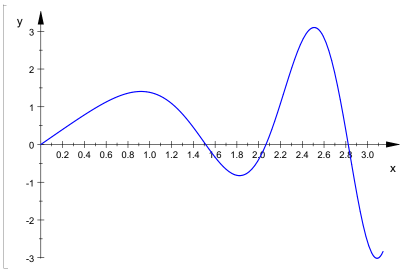
As an example, we will find the absolute maximum and minimum of f(x) = sin x + xcos(x^2) on the interval [0 , π].

```
reset();
```

```
f:=x->sin(x)+x*cos(x^2);
```

$$x \to \sin(x) + x\cos(x^2)$$

```
plot(f(x),x=0..PI);
```

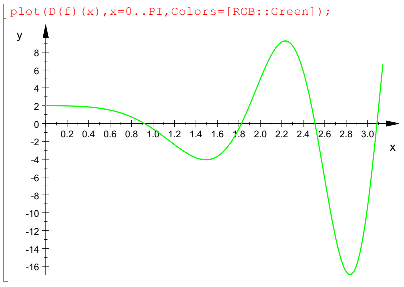


First use the above graph and the cursor to find approximate values of the absolute maximum and minimum.

Next, use the derivative to find exact values of the absolute maximum and minimum:

```
D(f);
```
$x \rightarrow \cos(x^2) + \cos(x) - 2\,x^2 \sin(x^2)$

```
plot(D(f)(x),x=0..PI,Colors=[RGB::Green]);
```



This function has a derivative at every point. Therefore the only critical points are the solutions of the equation f '(x) = 0.

```
solve(D(f)(x)=0,x);
```
$\text{solve}(\cos(x^2) + \cos(x) - 2\,x^2 \sin(x^2) = 0,\ x)$

MuPAD cannot find a general solution, so we will use the "fsolve" command to find decimal approximations to the solutions. From the graph, it is clear that there are four solutions of f ' (x) = 0, since the graph of f '(x) cuts the X-axis four times.

```
use(numeric, fsolve);
```

```
X[1]:=fsolve(D(f)(x)=0,x=0.8..1)[1][2];
```
0.9201095708

```
X[2]:=fsolve(D(f)(x)=0,x=1.6..2)[1][2];
```
1.824276689

```
X[3]:=fsolve(D(f)(x)=0,x=2.4..2.6)[1][2];
```
2.509682366

```
X[4]:=fsolve(D(f)(x)=0,x=3..PI)[1][2];
```
3.086995383

From the graph it is clear that each of the intervals specified in the above four commands contains exactly one zero of f (x).

Finally, we calculate the values of f at these four points and at 0 and π, the endpoints of the interval under consideration:

```
f(X[1]);f(X[2]);f(X[3]);f(X[4]);f(0);f(PI);
```
1.405270457

-0.824633074

3.100075094

-3.015500543

0

$\pi \cos(\pi^2)$

For comparison purposes, we calculate a decimal expansion for *f*(π):

```
float(f(PI));
```
-2.835869702

Therefore the absolute maximum is f (X[3]) = 3.100075094 and the absolute minimum is f(X[4]) = -3.015500543.

Instead of typing **"float(f(PI));"** we could have used the percent symbol that acts as a placeholder for the last value computed by MuPAD. For example,

```
f(PI);
```
$\pi \cos(\pi^2)$

Now use the percent symbol:

```
float(%);
```
-2.835869702

Derivatives and properties of graphs

The important characteristics of the graph of a function f(x) can be established by studying its first and second derivatives. These characteristics include the location of any local maxima, local minima, and points of inflection, and intervals in which the graph is increasing or decreasing, or is concave upward or concave downward.

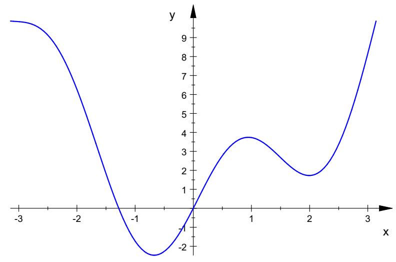
As an example we will study the function f (x) = $3\sin(2x) + x^2$, $-\pi \le x \le \pi$

```
reset();
```

```
f:=x->x^2+3*sin(2*x);
```
$x \rightarrow x^2 + 3\sin(2\,x)$

First we plot the graph of the function:

```
plot(f(x),x=-PI..PI);
```

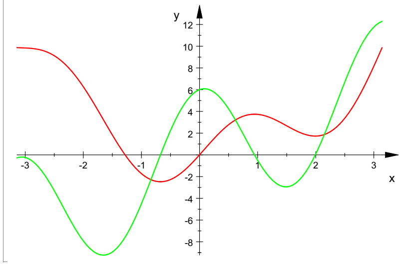


There appear to be two local minima, a local maximum close to x = 1, and possibly another local maximum near x = -3. At points where there is a local maximum or minimum the derivative is 0. We next compute the derivative and draw the graphs of f (x) and f '(x) on the same set of axes:

```
D(f);
```

$x \to 2\,x + 6\cos(2\,x)$

```
plot(f(x),D(f)(x),x=-PI..PI,Colors=[RGB::Red,RGB::Green]);
```



To find the exact locations of the local maxima and minima we solve the equation f '(x) = 0:

```
solve(D(f)(x)=0,x);
```

solve($2\,x + 6\cos(2\,x) = 0,\ x$)

Apparently, MuPAD does not know a general solution, so we will find the solutions using the "fsolve" command.

```
use(numeric,fsolve);
```

```
X[1]:=fsolve(D(f)(x)=0,x=-1..0)[1][2];
```

-0.6723755227

```
X[1]:=fsolve(D(f)(x)=0,x=0..1)[1][2];
```

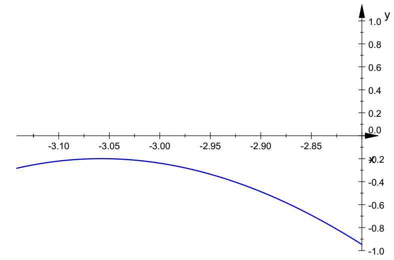
0.9457599482

```
X[1]:=fsolve(D(f)(x)=0,x=1..2)[1][2];
```

1.992913103

To see if there is another solution near -3, we zoom in on the graph of f '(x):

```
plot(D(f)(x),x=-PI..-2.8,ViewingBoxYRange=-1..1,
Colors=[RGB::Blue]);
```

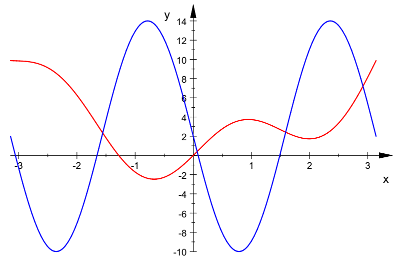


Since the graph of f '(x) does not touch the X-axis there is not an additional solution.
Note that, from the graph of f '(x), f '(x) is negative in the intervals (π , X[1]) and (X[2] , X[3]), and thus f (x) is decreasing on these intervals. f '(x) is positive in (x[1],X[2]) and (X[3] , π), and therefore f (x) is increasing there.
We now apply the First Derivative Test. Since f '(x) is negative to the left of X[1] and positive to the right, there is a local minimum at X[1]; similarly, there is a local minimum at X[3]. Since f '(x) is positive to the left of X[2] and negative to the right, there is a local maximum at X[2] .
The second derivative is used to find intervals of concavity and points of inflection. We will compute f "(x) and plot it and f (x) on the same set of axes.

```
D(D(f));
```

$x \to 2 - 12\sin(2\,x)$

```
plot((f(x),D(D(f))(x)),x=-PI..PI,Colors=[RGB::Red,RGB::Blue]);
```



Points of inflection occur at points where f "(x) = 0 and the second derivative changes sign. The graph is concave down on intervals where f "(x) < 0 and concave upward when f "(x) > 0; thus the points of inflection are the points where the concavity changes. Here there are four such points:

```
pi:=float(PI): Z[1]:=fsolve(D(D(f))(x)=0,x=-pi..(-3))[1][2];
```

$$-3.057868614$$

```
Z[2]:=fsolve(D(D(f))(x)=0,x=-2..-1)[1][2];
Z[3]:=fsolve(D(D(f))(x)=0,x=0..1)[1][2];
Z[4]:=fsolve(D(D(f))(x)=0,x=1..2)[1][2];
```

$$-1.654520366$$

$$0.08372403961$$

$$1.487072287$$

The graph of f(x) is concave up for x in ($-\pi$,Z[1]) , (Z[2],Z[3],), and (Z[4], π).
It is concave down in (Z[1],Z[2]) and (Z[3],Z[4]).

DIFFERENTIATION OF INVERSE FUNCTION

Given a function f, we wish to define a function g, called the inverse of f, which reverses the action of f, i.e., whenever f(a) = b, then g(b) = a. In order for this reversal process to define a function it is necessary that f be one-to-one: for each number b in the range of f there can be *only one* number a in the domain of f such that f(a) = b. If f is one-to-one no horizontal line can cut the graph of f more than once.

If g is the inverse of the one-to-one function f, then the graph of g is the set of points

{ (f (x) , x) | x in Dom(f)}.
Dom(g) = Ran(f) and Ran(g) = Dom(f), that g(f(x)) = x and f(g(x)) = x, that g is one-to-one with inverse f, and that the graph of g is the reflection of the graph of f in the line y = x.
If we differentiate the equation f(g(x)) = x using the chain rule we obtain the equation
f '(g(x))g '(x) = 1.
Solving this equation for g '(x) yields the formula for the derivative of the inverse g (x) of a function f (x) :
g' (x) = 1/f '(g(x)).

Now suppose that f(a) = b, and therefore that g(b) = a. From the last formula it follows that
g '(b) = 1/f '(g(b)) = 1/f '(a)
or
g'(f(a)) = 1/f '(a).
Since all of the numbers in the domain of g (and thus the domain of g ') are of the form f (a) for some a in the domain of f , it follows that the graph of g ' is the set of points
{ (f(a) , 1/f '(a)); a in dom f }.

MuPAD will use this representation to generate the graph of g'.

Example 1

Sometimes it is easy to find an explicit expression for the inverse g of a given function f by solving the equation f(y) = x for y. For example, suppose

$$f(x) = \frac{2\,x-3}{3\,x+7} .$$

```
reset();
```

```
f:=x->(2*x-3)/(3*x+7);
```

$$x \to \frac{2\,x-3}{3\,x+7}$$

To find g, the inverse of f, we interchange x and y and solve the resulting equation for y:

```
eq:=f(y)=x;
```

$$\frac{2\,y-3}{3\,y+7} = x$$

```
g:=solve(eq,y);
```

$$\begin{cases} \emptyset & \text{if } x = \frac{2}{3} \\ \left\{-\frac{7\,x+3}{3\,x-2}\right\} & \text{if } x \neq \frac{2}{3} \end{cases}$$

We can extract the formula from this case display as follows.

```
op(op(op(g,2),2),1)
```

$$-\frac{7\,x+3}{3\,x-2}$$

```
g:=g[2][1];
```

$$-\frac{7\,x+3}{3\,x-2}$$

We will verify the formula for g', i.e., we will show that g '(x) = 1/f '(g(x))
by computing both g'(x) and 1/f'(g(x)) and showing they are equal.
Note that we defined g as an expression in the variable x rather than using the *MuPAD* (variable-independent) function method; this was necessary because we were exchanging the variables x and y. To differentiate an expression in the variable x we use the following "diff" command:

```
diff(g,x);
```

$$\frac{3\,(7\,x+3)}{(3\,x-2)^2} - \frac{7}{3\,x-2}$$

```
simplify(%);
```

$$\frac{23}{(3\,x-2)^2}$$

Now we compute 1/ f '(g (x)):

```
diff(f(x),x);
```

$$\frac{2}{3\,x+7} - \frac{3\,(2\,x-3)}{(3\,x+7)^2}$$

```
subs(%,x=g);
```

$$\frac{3\left(\frac{2\,(7\,x+3)}{3\,x-2}+3\right)}{\left(\frac{3\,(7\,x+3)}{3\,x-2}-7\right)^2} - \frac{2}{\frac{3\,(7\,x+3)}{3\,x-2}-7}$$

```
simplify(%);
```

$$\frac{(3\,x-2)^2}{23}$$

```
1/(%);
```

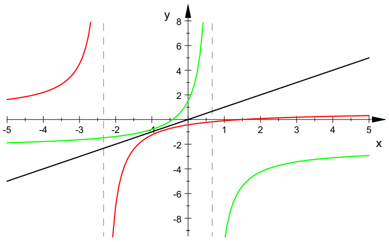
$$\frac{23}{(3\,x-2)^2}$$

We next plot the graphs of f, g, and the line y = x:

```
A:=plot::easy(f(x),x=-5..5,y=-5..5,Colors=[RGB::Red]):
B:=plot::easy(g(x),x=-5..5,y=-5..5,Colors=[RGB::Green]):
j:=x->x;
```

$$x \to x$$

```
C:=plot::easy(j(x), x=-5..5,y=-5..5,Colors=[RGB::Black]):
plot(A,B,C,Footer="Example 1: Figure 1");
```



Example 1: Figure 1

Note that the graph of g is the reflection of the graph of f in the line y = x.

Example 2.

```
reset():
```

Consider the function $f(x) = x + \sin\left(\frac{\pi x}{4}\right) - 2$; $0 \le x \le 8$.

The inverse of f will again be denoted by g.

In this case, as we will see below, we are not able to explicitly solve the equation f(y) = x for y = g(x), and thus generate a formula for g (x). However, we can still plot the graph of the inverse g and its derivative g ', and compute their values for numbers in their domains.

We first try to find a formula for g by solving f(y) = x, as in example 1.

```
f:=x->x-2+sin(PI*x/4);
```

$$x \to x - 2 + \sin\left(\frac{\pi x}{4}\right)$$

```
f(x);
```

$$x + \sin\left(\frac{\pi x}{4}\right) - 2$$

```
eq:=f(y)=x;
```

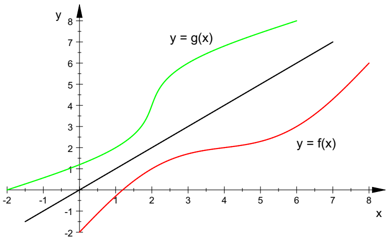
$$y + \sin\left(\frac{\pi y}{4}\right) - 2 = x$$

```
solve(eq,y);
```

$$\mathrm{solve}\left(y + \sin\left(\frac{\pi y}{4}\right) = x + 2,\ y\right)$$

Apparently *MuPAD* cannot solve this equation for y = g(x) in terms of x. However, even though we don't have a formula for g, we can still easily generate the graph of g since we know it consists of the set of points of the form (f (x) , x) for x in dom(f).

```
fGraph:=plot::easy([x,f(x)],x=0..8,Colors=[RGB::Red]):
T2:=plot::Text2d("y = f(x)",[6,2]):
gGraph:=plot::easy([f(x),x],x=0..8,Colors=[RGB::Green]):
T1:=plot::Text2d("y = g(x)",[2.5,7]):
identity:=plot::easy([x,x],x=-1.5..7,Colors=[RGB::Black]):
plot(fGraph,gGraph,identity,T1,T2,Footer="Example 2: Figure 1");
```



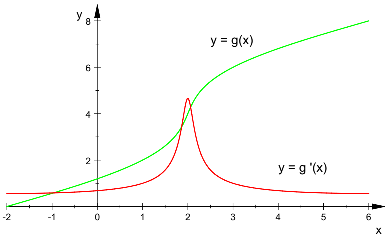
Example 2: Figure 1

Next, we plot g and g ' on the same set of axes. Recall that the graph of g ' is the set of points { (f (x) , 1/ f '(x)) ; x in dom f }.

```
fprime:=x->D(f)(x);
```

$$x \to f'(x)$$

```
DgGraph:=plot::easy([f(x),1/fprime(x)],x=0..8,Colors=[RGB::Red]):
T3:=plot::Text2d("y = g '(x)",[4,1.5]):
plot(gGraph,DgGraph,T1,T3,Footer="Example 2: Figure 2");
```



Example 2: Figure 2

Recall that g '(x) = 1/f '(g(x)) , $-2 \le x \le 6$.

```
gprime:=1/fprime(g(x));
```

$$\frac{1}{\frac{\pi \cos\left(\frac{\pi\, g(x)}{4}\right)}{4} + 1}$$

For example, we may find g' (2) by utilizing the fact that f (4) = 2 implies g (2) = 4.

```
subs(subs(gprime,g(x)=4),x=2); simplify(%);
```

$$\frac{1}{\frac{\pi \cos(\pi)}{4} + 1}$$

$$-\frac{4}{\pi - 4}$$

```
float(%);
```

4.659792366