Th.(Cayley-Hamilton) $P_A(A) = 0$

Given $A \in \mathbb{F}^{n \times n}$
Find $v \neq 0, \lambda : Av = \lambda v$
$\lambda-$ eigenvalue, $v-$eigenvector

1) λ is eigenvalue $\iff P_A(\lambda) = |\lambda I - A| = 0$
$\downarrow$
2) v is eigenvector $\iff 0 \neq v \in V_\lambda = N(A - \lambda I) = \{v : Av = \lambda v\}$

Algebraic multiplicity of λ_0 is max k s.t. $(\lambda - \lambda_0)^k$ divides $P_A(\lambda)$

Geometric multiplicity of λ_0 is $\dim V_{\lambda_0}$

Th.: $1 \leq$ Geo. $\leq$ Alge.

$A \in \mathbb{F}^{n \times n}$ is diagonalizable if:

$$\exists P : P^{-1}AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Th.:
1. eigenvectors of **different** eigenvalues are linearly independent.
2. $A \in \mathbb{F}^{n \times n}$ is diagonalizable $\iff$
   $\exists$ n linearly independent eigenvectors $\iff$
   $\exists$ an eigenvectors basis $\iff$
   $P_A(\lambda) = \prod_i (\lambda - \lambda_i)^{\alpha_i}$ and for all λ_i: Geo.=Alge. $\iff$
   $m_A(\lambda) = \prod_i (\lambda - \lambda_i)$

Note: $P = (v_1, \ldots v_n), v_i$ is eigenvector and λ_i its eigenvalue

Th.: For a normal matrix $A \in \mathbb{C}^{n \times n}$:
1. it is diagonalizable by unitary matrix, i.e.

$$\exists P \text{ unitary: } P^*AP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (P^* = P^{-1})$$

2. eigenvectors of **different** eigenvalues are orthogonal.

Def: for $A \in \mathbb{C}^{n \times n} : A^* \in \mathbb{C}^{n \times n}$ and $(A^*)_{i,j} = \bar{A}_{j,i}$
Def: $A \in \mathbb{C}^{n \times n}$ is **normal** if $AA^* = A^*A$.
Def: $P \in \mathbb{C}^{n \times n}$ is **unitary** if $PP^* = I$, i.e. $P^{-1} = P^*$.

$\Downarrow$

Th.: For a complex hermitian matrix $A^* = A$: $\forall i : \lambda_i \in \mathbb{R}$

$\Downarrow$

Th.: For a real $A \in \mathbb{R}^{n \times n}$ symmetric matrix $A^t = A$:
P is (*also*) real matrix and $P^{-1} = P^* = P^t$, i.e.,

$$\exists P : P^tAP = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \quad (P^t = P^{-1})$$