פתרון 4

1. פונקציות רציונליות: חשבו את האינטגרלים הבאים .

א. $\int \frac{3x+2}{x^2+5x+6}dx$ **ב.** $\int \frac{1}{(2x-1)(x+2)^2}dx$ **ג.** $\int \frac{2x-3}{x^2+7}dx$

ד. $\int \frac{x}{(x+1)(x^2+1)}dx$ **ה.** $\int \frac{x^3}{x^2+x+1}dx$ **ו.** $\int \frac{x-1}{x^2+2x+5}dx$ **ז.** $\int \frac{x}{x^3-1}dx$

א. $\int \frac{3x+2}{x^2+5x+6}dx$

$$\int \frac{3x+2}{x^2+5x+6}dx = \int \frac{3x+2}{(x+2)(x+3)} =_{(*)} \int \left(\frac{-4}{x+2}+\frac{7}{x+3}\right)dx = -4\ln|x+2|+7\ln|x+3|+c$$

(*) פירוק לשברים חלקיים:

$$\frac{3x+2}{(x+2)(x+3)} = \frac{A}{x+2}+\frac{B}{x+3} \rightarrow \quad 3x+2 = B(x+2)+A(x+3) \rightarrow \quad A=-4,\ B=7$$

ב. $\int \frac{1}{(2x-1)(x+2)^2}dx$

נפרק את השבר למנות חלקיות:

$$\frac{1}{(2x-1)(x+2)^2} = \frac{A}{2x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2} \quad \rightarrow 1 = A(x+2)^2 + B(x+2)(2x-1)+C(2x-1)$$

הציבו $x=-2$ וקבלו $1=-5C$ לכן $C=-\frac{1}{5}$, הציבו $x=\frac{1}{2}$ וקבלו A $1=\left(\frac{5}{2}\right)^2$ ולכן $A=\frac{4}{25}$,

הציבו $x=0$ וקבלו $1=4A-2B-C$ לכן $B=\frac{-2}{25}$. הציבו:

$$\int \frac{1}{(2x-1)(x+2)^2}dx = \int \frac{\left(\frac{4}{25}\right)}{2x-1}+\frac{\left(\frac{-2}{25}\right)}{x+2}+\frac{\left(\frac{-1}{5}\right)}{(x+2)^2}dx = \frac{4}{25}\ln|2x-1|-\frac{2}{25}\ln|x+2|+\frac{\frac{1}{5}}{x+2}+c$$

ג. $\int \frac{2x-3}{x^2+7}dx$

$$\int \frac{2x-3}{x^2+7}dx = \int \frac{2x}{x^2+7}dx - \int \frac{3}{x^2+7}dx = ln\left(x^2+7\right) - \frac{3}{\sqrt{7}}arctan\left(\frac{x}{\sqrt{7}}\right)+c$$

ד. $\int \frac{x}{(x+1)(x^2+1)}dx$

$$\int \frac{x}{(x+1)(x^2+1)}dx =_{(*)} \frac{1}{2}\int\left(\frac{-1}{x+1}+\frac{x+1}{x^2+1}\right)dx = -\frac{1}{2}\ln|x+1|+\frac{1}{4}\int\frac{2x+2}{x^2+1}dx =$$
$$-\frac{1}{2}\ln|x+1|+\frac{1}{4}\ln|x^2+1|+\frac{1}{4}\cdot 2\cdot arctg\, x + c$$

ה. $\int \frac{x^3}{x^2+x+1}dx$

$$\int \frac{x^3}{x^2+x+1}dx =_{(*)} \int\left(x-1+\frac{1}{x^2+x+1}\right)dx = \frac{1}{2}x^2 - x + \int\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx =$$
$$\frac{1}{2}x^2 - x + \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + c$$

(*) מעלת המונה גדולה ממעלת המכנה לכן יש לבצע חילוק פולינומים.

ו. $\int \frac{x-1}{x^2+2x+5}dx$

$$\int\frac{x-1}{x^2+2x+5}dx = \frac{1}{2}\int\frac{2x-2}{x^2+2x+5}dx = \frac{1}{2}\int\frac{2x+2-4}{x^2+2x+5}dx$$
$$= \frac{1}{2}\int\frac{2x+2}{x^2+2x+5}dx - \frac{1}{2}\int\frac{4}{x^2+2x+5}dx = \frac{1}{2}ln\left|x^2+2x+5\right| - 2\int\frac{1}{(x+1)^2+4}dx =$$
$$= \frac{1}{2}ln\left(x^2+2x+5\right) - \frac{2}{2}arctan\left(\frac{x+1}{2}\right) + c = \frac{1}{2}ln\left(x^2+2x+5\right) - arctan\left(\frac{x+1}{2}\right) + c$$

ז. $\int \frac{x}{x^3-1}dx$

$$\frac{x}{x^3-1} = \frac{x}{(x-1)(x^2+x+1)} = \frac{1/3}{(x-1)} + \frac{-1/3x+1/3}{(x^2+x+1)} = \frac{1/3}{(x-1)} - \frac{1}{3}\frac{x-1}{x^2+x+1}$$

$$\int\frac{x-1}{x^2+x+1}dx = \int\frac{\frac{1}{2}\cdot(2x+1)-\frac{1}{2}-1}{x^2+x+1}dx = \frac{1}{2}\int\frac{(2x+1)dx}{x^2+x+1} - \frac{3}{2}\int\frac{dx}{x^2+x+1} =$$
$$= \frac{1}{2}\int\frac{(2x+1)dx}{x^2+x+1} - \frac{3}{2}\int\frac{dx}{(x+\frac{1}{2})^2-\frac{1}{4}+1}$$

לכן:

$$\int \frac{x}{x^3-1}dx = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\left[\frac{1}{2}\int \frac{(2x+1)\,dx}{x^2+x+1} - \frac{3}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}\right] =$$

$$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) + \frac{1}{2}\cdot\frac{1}{\sqrt{3}/2}\,arctg\,\frac{x+\frac{1}{2}}{\sqrt{3}/2} + C = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) +$$

$$+ \frac{1}{\sqrt{3}}\,arctg\,\frac{2x+1}{\sqrt{3}} + C$$

2. חשבו את **האינטגרלים הטריגונומטריים** הבאים:

א. $\int \sin^2 x\cos^2 x\,dx$ **ב.** $\int \sin^3 x\cos^4 x\,dx$ **ג.** $\int \sin^4 x\,dx$

ד. $\int \frac{\cos^2 x}{\sin x}dx$ **ה.** $\int \frac{\cos^2 x}{\sin x}dx$

א. $\int \sin^2 x\cos^2 x\,dx$

$$\int \sin^2 x\cos^2 x\,dx = \int (\sin x\cos(x))^2 dx = \frac{1}{4}\int \sin^2(2x)\,dx = \frac{1}{4}\int \frac{1-\cos(4x)}{2}dx$$

$$= \frac{1}{8}\left(x - \frac{\sin(4x)}{4}\right) + C$$

ב. $\int \sin^3 x\cos^4 x\,dx$

$$\int \sin^3 x\cos^4 x\,dx = \int \sin^2 x\cos^4 x\sin x\,dx =_{u=\cos x,\quad du=-\sin x dx} -\int (1-u^2)u^4 du = \frac{-1}{5}u^5 + \frac{1}{7}u^7 + c =$$

$$\frac{-1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + c$$

ג. $\int \sin^4 x\,dx$

$$\int \sin^4 x\,dx = \int \left(\sin^2 x\right)^2 dx = \frac{1}{4}\int (1-\cos(2x))^2 dx = \frac{1}{4}\int \left(1-2\cos(2x)+\cos^2(2x)\right)dx$$

$$= \frac{1}{4}\int\left(1-2\cos(2x)+\frac{\cos(4x)+1}{2}\right)dx = \frac{1}{4}\left(x-\sin(2x)+\frac{\sin(4x)}{8}+\frac{x}{2}\right)+C$$

$$= \frac{1}{4}\left(-\sin(2x)+\frac{\sin(4x)}{8}+\frac{3x}{2}\right)+C$$

ד. $\int \frac{\cos^2 x}{\sin x}dx$

$$\int \frac{\cos^2 x}{\sin x}dx \underset{\substack{t=\cos(x) \\ dt=-\sin(x)dx}}{=} \int \frac{t^2}{-\sin^2 x}dt = \int \frac{t^2}{t^2-1}dt = \int \frac{t^2-1+1}{t^2-1}dt$$

$$= \int 1+\frac{1}{t^2-1}dt = \int 1+\frac{1/2}{t-1}-\frac{1/2}{t+1}dt$$

$$= t+\frac{\ln(|t-1|)}{2}-\frac{\ln(|t+1|)}{2}+C$$

$$= \cos(x)+\frac{\ln(|\cos(x)-1|)}{2}-\frac{\ln(|\cos(x)+1|)}{2}+C$$

ה. $\int \frac{1}{\sin x+\cos x}dx$

נציב: $u=\tan\frac{x}{2},\ \sin x=\frac{2u}{1+u^2},\ \cos x=\frac{1-u^2}{1+u^2},\ dx=\frac{2}{1+u^2}du$

ונקבל:

$$\int \frac{1}{\sin x+\cos x}dx = \int \frac{-2}{u^2-2u-1}du$$

נפרק לשברים חלקיים ונפתור את האינטגרל:

$$\int \frac{-2}{u^2-2u-1}du = \frac{1}{\sqrt{2}}(\ln(\sqrt{2}\tan(\frac{x}{2})+\sqrt{2}+2)-\ln(\sqrt{2}\tan(\frac{x}{2})-\sqrt{2}-2))$$